Cloud and precipitation Droplet fall speed

The terminal velocity of an object falling through a fluid is usually determined by a force balance between gravity and some drag force trying to slow the object down. The drag force depends on the velocity of the particle and therefore can be solved for when the forces are equal and opposite. The gravitational force is a so-called body force referring to the fact that the force acts on the entire body of the object. The drag force is a surface force where the force acts on the surface area of the object. There are two classes of drag force balance against the downward force of gravity:

1. Diffusive or viscous
    - The surface drag force acts (more or less) along the sides of the falling object and the force is due to the fluid fighting against the velocity shear along the sides of the object
    - There are two forms of viscous force depending on the source of the viscosity:
        a. Stokes drag: molecular viscosity dominates the viscosity
        b. Eddy drag: where the drag associated with eddies dominates the viscosity
2. Dynamic pressure
    - The drag force is the result of the molecules in front of the object colliding with the object

Terminal velocity and Shear viscosity

The viscous drag can be explained as follows. An object of radius, r, falls through some fluid.   The viscous force acts on the sides of the sphere through shear stress which has the units of force per unit area like pressure. As a result, some of the momentum of the object diffuses into the surrounding fluid accelerating the fluid and slowing the descent of the object. The flux of momentum due to shear is defined as the shear stress, τ.

$$\vec{\tau} = \vec{F}_{mom} = \upsilon \frac{\partial \rho_{mom}}{\partial r} \hat{z} = \upsilon \rho_{fluid} \frac{\partial w}{\partial r} \hat{z} = \mu_{fluid} \frac{\partial w}{\partial r} \hat{z} \tag{1}$$

Its units are $m^2 \ s^{-1} \ kg \ m^{-3} \ m \ s^{-1} \ m^{-1} = kg \ m \ s^{-1} \ m^{-2}$ which is force per unit area or pascals, the same units as pressure.

We integrate this stress over the portion of the area of the sphere that experiences drag due to shear and set it equal and opposite to the gravitational force on the object to determine the viscous terminal velocity.

$$\vec{f}_{drag} + \vec{f}_{grav} = 0 \tag{2}$$

$$\vec{f}_{drag} = \int \vec{\tau} dA = \int \mu_{fluid} \frac{\partial w}{\partial r} \hat{z} \ dA \approx \mu_{fluid} \frac{\partial w}{\partial r} A\hat{z} \approx \mu_{fluid} \frac{w}{r} a4\pi r^2 \hat{z} \tag{3}$$

where a is a scale factor representing the fraction of the area feeling this viscous shear force. The gravitational force on the falling object is

$$\vec{f}_{grav} = -\int (\rho_{obj} - \rho_{fluid}) g dV \ \hat{z} = -\int (\rho_{obj} - \rho_{fluid}) g 4\pi r^2 dr \ \hat{z} = -g \frac{4}{3} \pi r^3 (\rho_{obj} - \rho_{fluid}) \ \hat{z} \tag{4}$$

where the last equations are true only for spheres. Again we have assumed the fluid is in hydrostatic equilibrium such that only the fraction of the density that is larger than that of the

fluid cases the object to accelerate downward. We set the drag and gravitational forces to be equal and get

$$-g\frac{4}{3}\pi r^3\left(\rho_{obj}-\rho_{fluid}\right)\hat{z}=a\mu_{fluid}\frac{w}{r}4\pi r^2\hat{z} \tag{5}$$

We then solve for the terminal velocity for viscous conditions

$$w_{term}=-\frac{1}{3a}\frac{g}{\upsilon_{fluid}}\frac{\left(\rho_{obj}-\rho_{fluid}\right)}{\rho_{fluid}}r^2 \tag{6}$$

where *a* can now be thought of as including the shape dependence to deal with the geometric dependence of the shape of the falling object. *a* is generally less than one indicating the fraction of the surface area that the viscous stress works on. (6) is known as Stokes drag.

Eddy viscosity

For a fluid with well defined viscosity, we can apply (6) directly to calculate the terminal velocity. For air, this is a bit tricky because the molecular viscosity of air is very small and will likely not be the limiting drag force for a macroscopic water droplet. The falling droplet will create eddies around the droplet. These eddies will have an eddy viscosity associated with them that will transfer momentum from the falling droplet to the surrounding air. We can estimate this effect crudely as follows. We know diffusivity, which is analogous to kinematic viscosity, is equal to a length times a velocity. A simple approximate viscosity is therefore

$$\upsilon_{eddy}\sim 2rw \tag{7}$$

where the factor of 2 is simply to use the diameter of the droplet because it is likely a more relevant scale than the radius of the droplet. Plugging this into (6) yields

$$w_{term}=-\frac{1}{3a}\frac{g}{\upsilon_{eddy}}\frac{\left(\rho_{obj}-\rho_{fluid}\right)}{\rho_{fluid}}r^2=-\frac{1}{3b}\frac{g}{2rw_{term}}\frac{\left(\rho_{obj}-\rho_{fluid}\right)}{\rho_{fluid}}r^2 \tag{8}$$

where the factor of *b* accounts for the approximations in (7) and generalizing the geometry of an object beyond a sphere.

$$w_{term}^2=\frac{1}{6b}gr\frac{\left(\rho_{obj}-\rho_{fluid}\right)}{\rho_{fluid}} \tag{9}$$

$$\vec{w}_{term}=-\left[\frac{1}{6b}gr\frac{\left(\rho_{obj}-\rho_{fluid}\right)}{\rho_{fluid}}\right]^{1/2}\hat{z} \tag{10}$$

Based on measured fall speed of raindrops, *b* is ~0.1 which represents some combination of either over estimating the actual eddy viscosity and a smaller effective surface area affected by the eddy viscosity. So the terminal velocity's dependence on *r* in a low viscosity fluid in (10) is quite different from that in a high viscosity fluid (6).

We can plug in values in (6) and (10) for air and liquid water droplets to better understand the terminal velocity. Using $\upsilon_{air}\sim$ 2e-5 m^2/s, $\rho_{air}\sim$ 1 kg/m^3, $\rho_{obj}\sim$ 1e3 kg/m^3, and *r* =1 mm, the two terminal velocities are 163 m/s and 1.3 m/s using the molecular viscosity and

eddy viscosity of air respectively. Clearly the eddies exert far more drag than the molecular viscosity of air at the scale of raindrops.

Stagnation pressure

Another drag effect is so-called stagnation pressure which is associated with the fact that molecules of the fluid directly in the path of the object essentially must get accelerated to the fall velocity of the object via a collision with the object. The collisions with the fluid molecules yield an upward force on the falling object that is essentially a pressure. In fact this force is called a dynamic (as opposed to a hydrostatic) pressure. This pressure is approximately

$$P_{dyn} \sim \frac{1}{2}\rho_{fluid}w^2\ \hat{z} \tag{11}$$

So the upward force of the object due to the dynamic pressure is the stagnation pressure times the crossectional area of the object.

$$\vec{f}_{stag} = A\vec{P}_{dyn} \sim c\frac{\pi}{2}r^2\rho_{fluid}w^2\ \hat{z} \tag{12}$$

where c is another scale factor like a and b to account for the actual aeefective area on which the dynamic pressure works. Setting (4) equal to (12) yields

$$c\frac{\pi}{2}r^2\rho_{fluid}w^2 = -g\frac{4}{3}\pi r^3\left(\rho_{obj} - \rho_{fluid}\right) \tag{12}$$

$$w^2 = g\frac{2}{3c}r\frac{\left(\rho_{obj} - \rho_{fluid}\right)}{\rho_{fluid}}$$

$$w = -\left[\frac{2}{3c}gr\frac{\left(\rho_{obj} - \rho_{fluid}\right)}{\rho_{fluid}}\right]^{1/2}\ \hat{z} \tag{13}$$

Based on measurements, $c \sim 0.1$. Note that (13) looks very similar to (10) but was derived via a conceptually different physical forces associated with the object falling through the air.

The upward force experienced by the droplet is actually the sum of these forces.

fall speed of water droplets
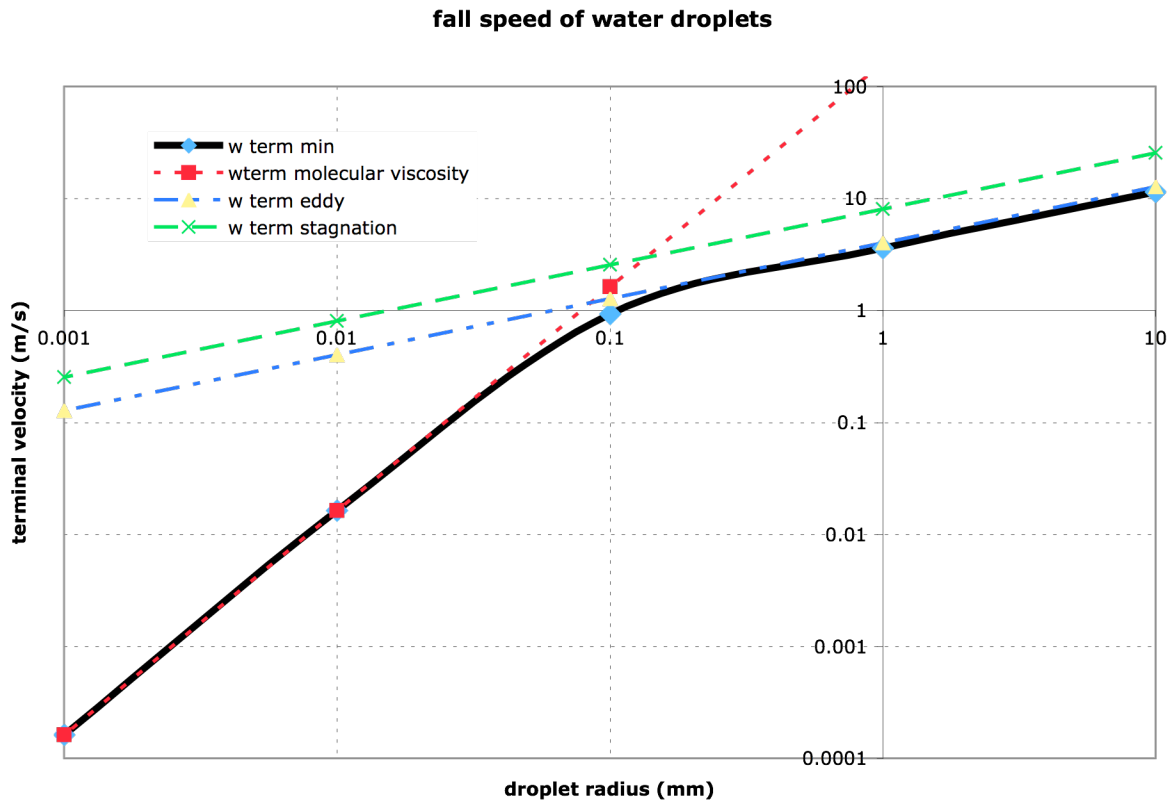
The plot shows the approximate fall speeds over a wide range of droplet sizes for typical sea level conditions. Note that the molecular viscosity depends on the density of the air. At higher altitudes the molecular diffusivity increase because the mean free path between molecular collisions increases at lower densities.

At smaller, cloud droplet sizes, the dominant drag force is due to molecular viscosity. At larger, raindrop scales, the drag force is due to combined eddy viscosity and stagnation pressure. The transition between the two types of behavior occurs near a droplet radius of 200 microns.

For typical cloud droplet sizes of ~10 microns or less the terminal velocity is quite small, 1 cm/s or less. Reaching a 1 cm/s updraft speed requires very little acceleration. The fractional temperature difference relative to surrounding air required to accelerate an air parcel to 1 cm/s after only 1 m of acceleration is only $\Delta w^2/(g\Delta z) = 10^{-4}/10 = 10^{-5}$ which is equal to a temperature difference of ~ 2.5 mK. It's no wonder clouds are turbulent given the temperature changes due to latent heat transfer via condensation and evaporation.

$$\Delta v = \sqrt{a\Delta z} = \left(-g\left[\frac{T'-T}{T}\right]\Delta z\right)^{1/2}$$

$$\left[\frac{T'-T}{T}\right] = -\frac{\Delta v^2}{g\Delta z}$$

So cloud droplets don't fall much and therefore largely stay suspended in the air. Raindrops which are ~1 mm or larger with terminal velocities of several m/sec do fall out of clouds except in sufficiently strong updrafts. An updraft 1K warmer than the environment needs 250 m of acceleration to reach velocities of several m/sec that are sufficient to hold a 1 mm droplet aloft.

Nonspherical droplets and polarized weather radars

Nonspherical "drops" like ice crystals are of course more complicated and won't be treated in any detail here. Ice crystals like snow flakes can fall much more slowly that predicts above because they lie flat such that the stagnation pressure is much higher for a given mass particle because they have such large surface area on that side.

It is also worth noting that the larger raindrops flatten as they fall so that they are no longer spheres. In fact droplets larger than about 6 mm radius tend to break up as they fall limiting the size of liquid hydrometeors. Polarized weather radars take advantage of the flattening effect to measure rain rate from the backscatter as the radar signals bounce off the hydrometeors and return to the radar. By using both horizontally and vertically polarized signals, the horizontal signal "sees" a bigger droplet than the vertically polarized signal and so the backscattered horizontally polarized signal is stronger than the backscattered vertically signal and this difference in backscatter strength is used to get a better estimate of the rainfall rate.

The reason is the rainfall rate is the flux of the rain drops

$$F_{drops} = \rho_{drops}\, w_{drops} = n_{drops}\, m_{drop}\, w_{drop}.$$

The mass of a spherical droplet is $4/3\, \pi\, r^3\, \rho_L$. The fall speed of a raindrop as we have just seen is proportional to $r^{1/2}$. So the rain rate is proportional to $r^3\, r^{1/2} = r^{7/2}$. The problem is the radar backscatter from a single polarization is proportional to r^6 which is quite different from $r^{7/2}$ and means that the backscatter from a few very large droplets can completely dominate the radar backscatter and give an unrepresentative estimate of rain rate. The differential backscatter between the horizontal and vertical polarizations scales more like r^4 which is much closer to $r^{7/2}$ and therefore allows much better rainfall rates to be estimated from the ("S-pol") radars.