1. *A rising air parcel in the core of a thunderstorm achieves a vertical velocity of 80 m/s (similar to the midterm) when it reaches a neutral buoyancy altitude at approximately 12 km and 200 mb. Assume the background atmosphere is isothermal at this altitude. How much will the air parcel overshoot the neutral buoyancy level altitude?*

   a. *Set the kinetic energy/mass of the air parcel at the neutral buoyancy level equal to the work/mass done by the negative acceleration of the air parcel above the neutral buoyancy altitude. The acceleration is proportional to the Brunt Vaisala frequency squared: Eq's 37 & 38 from adiabatic lapse rate notes.*

$$a = \frac{d^2x}{dt^2} = -\omega^2 x = -x\left(\frac{g}{\theta}\frac{\partial\theta}{\partial z}\right) \qquad (37\&38)$$

$$\frac{\partial\theta}{\theta\,\partial z} = -\frac{\left[\frac{\partial T_{adiabatic}}{\partial z} - \frac{\partial T_1}{\partial z}\right]}{T_1} \qquad (32)$$

$$\frac{1}{2}w^2 = -\int_0^{\Delta z} a\,dz = \int_0^{\Delta z}\omega^2 z\,dz = \omega^2\int_0^{\Delta z} z\,dz = \omega^2\frac{\Delta z^2}{2}$$

   b. *Solve for the height above the neutral buoyancy level at which the potential energy per unit mass equals that of the kinetic energy and therefore at which the parcel stops moving vertically.*

$$\Delta z = \frac{w}{\omega} = \frac{w}{\left(\frac{g}{\theta}\frac{\partial\theta}{\partial z}\right)^{1/2}} = w\left(\frac{T_1}{g\left[\frac{\partial T_1}{\partial z} - \frac{\partial T_{adiabatic}}{\partial z}\right]}\right)^{1/2}$$

In this case, d T_1/dz = 0 and $T_1 \sim$ 220K and w = 80 m/s^2.

$$\Delta z = 80\left(\frac{220}{10[0.01]}\right)^{1/2} = 80\left(\frac{220}{10[0.01]}\right)^{1/2} = 3833m$$

This is quite a bit of overshoot and can be important for instance exchange between the stratosphere and troposphere. Such large overshoot requires a lot of CAPE which can only occur over land where a lot of low level moisture is available with cooler air aloft.

2. *N_2 collisional crosssection (you'll need this for problem 3)*
   a. *Given that the diffusivity of air for typical surface conditions is 2e-5 m^2/s, work backwards to determine the approximate collisional crosssectional area of N_2 molecules colliding with other N_2 molecules*

$$D = \sqrt{\frac{8}{\pi m}} \frac{(k_B T)^{3/2}}{A_c P} \tag{30}$$

$$A_c = \sqrt{\frac{8}{\pi m}} \frac{(k_B T)^{3/2}}{D P}$$

check units: $\frac{kg^{3/2} m^3 s\, s^2\ m^2}{kg^{1/2} s^3 m^2 kg\ m} = m^2$

m = 0.028 kg/mole /N_A = 4.65e-26 kg/molecule
k_B = 1.38e-23 J/K
D = 2.2e-5 m^2/s
P = 1000 mb
T = 288 K
A_c = 8.4e-19 m^2.

3. *When the mean free path equals a scale height, molecules can potentially overcome earth's gravity and escape to space.*

   a. *Determine the approximate pressure level at which this occurs in Earth's atmosphere?*

$$\lambda = \frac{1}{A_c N} \tag{1}$$

Pressure scale height

$$H = \frac{k_B T}{mg} \tag{2}$$

$$\frac{1}{A_c N} = \frac{k_B T}{mg} = \frac{k_B T}{A_c P} \tag{3}$$

$$P = \frac{mg}{A_c} \tag{4}$$

Using a mass of 28 g/mole = 4.7e-26 kg, yields a pressure of 4.5e-7 Pa = 4.5e-9 mb.

b. Determine the approximate height of the exobase for N_2.
Assume the average temperature of the atmosphere up to 90 km is 230K and the air is sufficiently mixed that the mean molecular mass is 28.96 g/mole. Above 90 km, assume the average temperature is 500 K and the scale height depends on the selected molecule, in this N_2.

The number of pressure scale heights between the surface and 4.5e-9 mb is ln(1000/4.5e-9) ~ 26. Assuming an average temperature of 230 K below 90 km and a mean molecular mass of 28.96 g/mole and a surface pressure of 1000 mb yields a pressure at 90 km of approximately 0.0016 mb [$P = P_{surf}$ exp(-90/H)] and there are approximately 13.4 scales heights between the surface and 90 km [=ln(1000 mb/0.0016 mb)]. This means another 12.7 scale heights (=26-13.4) are needed to reach the exobase. Assuming a temperature of 500 K and using the N_2 mass, the average scale height above 90 km is 15.2 km. The exobase altitude for N_2 is therefore approximately 12.7*15.2 km + 90 km = 280 km.

4. *Using the form where* $u = \bar{u} + u'$,
   a. *Show that the horizontal moisture flux is* $\rho_a \overline{uq} = \rho_a \bar{u}\,\bar{q} + \rho_a \overline{u'q'}$

First key point: when we decompose a variable, x, into its mean and perturbations, the mean of the perturbations, $\overline{x'} = 0$.
Now lets calculate the horizontal moisture flux.

$$\rho_a \overline{uq} = \rho_a \left(\overline{[\bar{u} + u'][\bar{q} + q']}\right) = \rho_a \left(\overline{\bar{u}\bar{q} + \bar{u}q' + u'\bar{q} + u'q'}\right)$$

$$\rho_a \overline{uq} = \rho_a \left(\overline{\bar{u}\bar{q}} + \bar{u}\,\overline{q'} + \overline{u'}\,\bar{q} + \overline{u'q'}\right) = \rho_a \bar{u}\,\bar{q} + \rho_a \overline{u'q'}$$

The reason for the last step is $\overline{q'} = 0$ and $\overline{u'} = 0$.

   b. *Describe the kinds of fluxes that the two terms,* $\rho_a \bar{u}\,\bar{q}$ *and* $\rho_a \overline{u'q'}$ *represent.*

The first is just the mean wind times the mean specific humidity which is simple mean advection. The second term is an eddy correlation term which is nonzero if and only if the variations in velocity are correlated (or anticorrelated) with the variations in specific humidity.

   c. *Explain why for the vertical flux* $\rho_a \overline{wq} = \rho_a \bar{w}\,\bar{q} + \rho_a \overline{w'q'} \cong \rho_a \overline{w'q'}$

There is typically not much of an average vertical velocity because you can't accumulate mass in the upper troposphere. As some mass goes up, other mass has to come down.

5. SH Flux into surface
Consider a citrus tree grove where the trees are about 5 m in height and the wind at 10 m is blowing at 2 m/sec with a surface temperature of 0^oC and pressure of 1000 mb.
a. Calculate an approximate eddy diffusivity
Assume the temperature at 10 m is 1^oC warmer than the surface temperature

We'll crudely assume the scale of the eddies is the scale of the trees and we'll use the wind at 5 m height as the eddy velocity which we'll take as half the wind at 10 m height. So the eddy diffusivity is approximately D ~ 5 m 2 m/s ½ = $5m^2/s$.

b. Calculate the vertical sensible heat flux

The diffusive flux is

$$F_{dif} = -D\,\rho\,C_p\,dT/dz$$

$$\sim -5m^2/s\ 1.2\ kg/m^3\ 1004\ J/kg/K\ 1/10\ K/m$$

$$= -\ 600\ Wm^2$$

This is large and is probably a bit of an overestimate because the diffusivity is probably overestimated a bit.

c. Is it up or down?

Diffusion works in the opposite direction as the (temperature) gradient. The temperature increases upwards so the sensible energy flux is directed downwards in the atmosphere.

6. Surface evaporative flux increase with global warming
Suppose the Earth's surface were to warm by 4^oC as predicted by some climate models. Suppose that the relative humidity of the air and the winds were to remain the same. Using the aerodynamic formula for latent heat flux, determine the ratio of the new surface latent heat flux to the present surface latent heat flux.

The aerodynamic surface latent heat flux is

$$F_{LH} = \rho_a L_v C_W U(q_s - q_a)$$

and the ratio of the new to the present (or old) latent heat flux is

$$\frac{F_{LH-new}}{F_{LH-old}} = \frac{\rho_{a-new} L_v C_{W-new} U_{new}(q_{s-new} - q_{a-new})}{\rho_{a-old} L_v C_{W-old} U_{old}(q_{s-old} - q_{a-old})}$$

We expand the specific humidity ratio in terms of what the specific humidity is

$$\frac{(q_{s-new}-q_{a-new})}{(q_{s-old}-q_{a-old})}=\frac{\left(\frac{m_v e_{s-new}}{(m_d P_{s-new}-m_v e_{s-new})}-\frac{m_v e_{a-new}}{(m_d P_{a-new}-m_v e_{a-new})}\right)}{\left(\frac{m_v e_{s-old}}{(m_d P_{s-old}-m_v e_{s-old})}-\frac{m_v e_{a-old}}{(m_d P_{a-old}-m_v e_{a-old})}\right)}$$

$$\frac{(q_{s-new}-q_{a-new})}{(q_{s-old}-q_{a-old})}\cong\frac{\left(\frac{m_v e_{s-new}}{(m_d P_{s-new})}-\frac{m_v e_{a-new}}{(m_d P_{a-new})}\right)}{\left(\frac{m_v e_{s-old}}{(m_d P_{s-old})}-\frac{m_v e_{a-old}}{(m_d P_{a-old})}\right)}\cong\frac{\left(\frac{m_v e_{s-new}}{(m_d P_{s-new})}-\frac{m_v e_{a-new}}{(m_d P_{s-new})}\right)}{\left(\frac{m_v e_{s-old}}{(m_d P_{s-old})}-\frac{m_v e_{a-old}}{(m_d P_{s-old})}\right)}$$

$$\frac{(q_{s-new}-q_{a-new})}{(q_{s-old}-q_{a-old})}\cong\frac{\left(\frac{m_v e_{s-new}-m_v e_{a-new}}{(m_d P_{s-new})}\right)}{\left(\frac{m_v e_{s-old}-m_v e_{a-old}}{(m_d P_{s-old})}\right)}$$

Now we use $e = e_s\, RH$ and the differential form of the Clausius Clapeyron equation

$$\frac{de_s}{e_s}=\frac{L_v}{R_v T^2}dT$$

So when RH is constant,

$$\frac{de}{e}=\frac{d(RH\ e_s)}{RH\ e_s}=\frac{RH\ d(e_s)}{RH\ e_s}=\frac{de_s}{e_s}=\frac{L_v}{R_v T^2}dT$$

So

$$\frac{e_{new}}{e_{old}}=\frac{(RH\ e_s)_{new}}{(RH\ e_s)_{old}}=\frac{(e_s)_{new}}{(e_s)_{old}}=\frac{(e_s)_{old}+\Delta e_s}{(e_s)_{old}}=\frac{(e_s)_{old}+(e_s)_{old}\frac{L_v}{R_v T^2}\Delta T}{(e_s)_{old}}=1+\frac{L_v}{R_v T^2}\Delta T$$

so

$$\frac{(q_{s-new}-q_{a-new})}{(q_{s-old}-q_{a-old})}\cong\frac{\left(\frac{m_v e_{s-new}-m_v e_{a-new}}{(m_d P_{s-new})}\right)}{\left(\frac{m_v e_{s-old}-m_v e_{a-old}}{(m_d P_{s-old})}\right)}=\frac{\left(\frac{RH_s e_{s-s-new}-RH_a e_{s-a-new}}{(P_{s-new})}\right)}{\left(\frac{RH_s e_{s-s-old}-RH_a e_{s-a-old}}{(P_{s-old})}\right)}$$

$$\frac{(q_{s-new}-q_{a-new})}{(q_{s-old}-q_{a-old})}\cong\frac{P_{s-old}}{P_{s-new}}\left(\frac{RH_s e_{s-s-old}\left(1+\frac{L_v}{R_v T^2}\Delta T\right)-RH_a e_{s-a-old}\left(1+\frac{L_v}{R_v T^2}\Delta T\right)}{RH_s e_{s-s-old}-RH_a e_{s-a-old}}\right)$$

$$\frac{(q_{s-new}-q_{a-new})}{(q_{s-old}-q_{a-old})}\cong\frac{P_{s-old}}{P_{s-new}}\left(\frac{RH_s e_{s-s-old}-RH_a e_{s-a-old}}{RH_s e_{s-s-old}-RH_a e_{s-a-old}}\right)\left(1+\frac{L_v}{R_v T^2}\Delta T\right)=\frac{P_{s-old}}{P_{s-new}}\left(1+\frac{L_v}{R_v T^2}\Delta T\right)$$

Notice that with these simplifications we didn't even need to know what the current relative humidity is. We simply had to assume it remains unchanged in a warmer climate. This is approximately what the global climate models do. (Is this correct? I don't know. There are some observations that are consistent with this type of constant RH behavior with warming).

We then assume the surface pressure does not change with global warming to simplify this further. There actually will be a slight increase in surface pressure as the amount of water vapor in the atmosphere increases but we ignore that subtle change. So

$$\frac{(q_{s-new} - q_{a-new})}{(q_{s-old} - q_{a-old})} \cong \left(1 + \frac{L_v}{R_v T^2}\Delta T\right)$$

We also need to consider potential changes in $\rho_{a-new} L_v C_{W-new} U_{new}$. We will ignore changes in the latent heat although there will be a slight decrease as the temperature changes (the water molecules are more active when they are warmer and therefore it is easier for them to fly off the surface. Therefore there is not as much energy associated with them changing phase and flying off the surface as temperature increases). The temperature dependence of themolar latent heat, L_m, is

$$L_m = \alpha\ (T_{crit} - T)^{0.375}$$

where T_{crit} is the critical temperature which is the temperature where the distinction between vapor and liquid ceases to exist (647.096 K for water) and α is a constant that depends on the liquid. From this we see that

$$\frac{L_{m-new}}{L_{m-old}} = \frac{(T_{crit} - T_{new})^{0.375}}{(T_{crit} - T_{old})^{0.375}} = \frac{(647.096 - 292)^{0.375}}{(647.096 - 288)^{0.375}} = 0.996$$

We will ignore any possible changes in the wind speed. In some general sense the heat engine concept says there should be some increase in the winds with global warming but we ignore this. The drag coefficient does not change if the surface roughness does not change. If the plants start changing in a given area due to global warming then there could be a change. Over the oceans, C_w depends on the wnds speed which creates surface roughness over the oceans. We have assumed the winds don't change so C_w over the oceans won't change.

The density of the air changes slightly

$$\frac{\rho_{air-new}}{\rho_{air-old}} = \frac{P_{surf-new}}{P_{surf-old}}\frac{T_{old}}{T_{new}} = \frac{T_{old}}{T_{new}}$$

Plugging these last two equations into the surface latent heat flux ratio equation yields

$$\frac{F_{LH-new}}{F_{LH-old}} = \frac{\rho_{a-new} L_v C_{W-new} U_{new} (q_{s-new} - q_{a-new})}{\rho_{a-old} L_v C_{W-old} U_{old} (q_{s-old} - q_{a-old})} = \frac{T_{old}}{T_{new}}\left(1 + \frac{L_v}{R_v T^2}\Delta T\right)$$

So for a 4 K increase in surface temperature, the latent heat flux will increase by 24%.

$$\frac{F_{LH-new}}{F_{LH-old}} = \frac{288}{292}\left(1 + \frac{2.5e6}{462 * 288^2}4\right) = 1.24$$

The present globally averaged latent heat flux is 78 W/m^2. So a 24% increase would raise it to 97 W/m^2.

So even though water vapor is a greenhouse gas that enhances and roughly doubles the global warming due to increased CO_2 concentrations, the increase in the surface latent heat flux with global warming will actually cool the Earth's surface. Roughly half of the increase in downward IR flux into the surface is offset by an increase in the latent heat flux from the surface. The other half is made up primarily by the increase in IR emitted by the surface due to the higher surface temperatures.

7. *Diffusion scaling: The time to cook a hard-boiled egg is ~12 minutes. Based on your understanding of diffusion (see eq. (16) of the diffusion lecture), how long should it take to cool a watermelon in a refrigerator down to the refrigerator's temperature. State whatever assumptions you need to make.*

$$\tau_\lambda = n\tau_\lambda = n\frac{\lambda}{v_t} = \left(\frac{X}{\lambda}\right)^2 \frac{\lambda}{v_t} = \frac{X^2}{v_t \lambda} \tag{16}$$

The point here is that the mean free path and velocity of molecules in the egg and in the watermelon are about the same. The watermelon is much larger meaning that X is much larger and the time to cook should therefore be about the ratio of their linear dimension size squared. If the watermelon is about 5 times larger than the egg in each dimension then the watermelon will take about 25 times as long to cool down.

One can do this more precisely by accounting for the fact that the velocities in the egg will be higher because of the higher temperatures near boiling in the egg and the cooler temperatures in the watermelon near freezing. The ratio of the velocities (ignoring differences in the masses of the average molecules which are both probably primarily water) will be

$$v_{egg}/v_{watermelon} = (T_{egg}/T_{watermelon})^{1/2} = (373/273)^{1/2} = (1.37)^{1/2} = 1.17.$$

So the velocity effect is there but the sensitivity to the square of the relative sizes of the objects is much larger. So taking into account the velocity dependence on temperature, the watermelon that is 5 times larger than the egg in terms of their linear dimensions will take about 1.17 times 25 which is 29 times as long to cool as the egg will take to cook.

Since the egg takes 12 minutes or 1/5 of an hour, the watermelon will take 5-6 hours to cool down.

You can use this physics cookbook to figure out how long your Christmas turkey, ham or roast beast will take to cook.