Does the matrix satisfy $Ax = 0$?

$$\begin{bmatrix} 1 & 1 & 4 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 3 & 2 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By using matrix multiplication, get . .

$$\left.\begin{aligned} x_1 + x_2 + 4x_3 &= 0 \\ x_1 + 2x_2 + 3x_3 &= 0 \\ x_1 + 3x_2 + 2x_3 &= 0 \\ x_1 + 4x_2 + x_3 &= 0 \end{aligned}\right\} \text{Solve this system}$$

(or equivalently ...) Use Gaussian elimination:

- ① Subtract row 1 from rows 2-4 * then,
- ② Subtract 2 × row 2 from row 3
- ③ Subtract 3x row 2 from row 4.

Step 1

$$\begin{bmatrix} 1 & 1 & 4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 3 & -3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \end{bmatrix} = ?$$

Step 2 & 3

$$\begin{bmatrix} 1 & 1 & 4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \end{bmatrix} =$$

After ~~simplify~~ simplifying the equations,

$$x_2 = x_3$$
$$x_1 = -5x_2$$

Hence $Ax = 0$ for:

$$x = \begin{bmatrix} -5 \\ 1 \\ 1 \end{bmatrix}$$ or any multiple thereof.

In other words, the 1^{st} column in A is $\frac{1}{5}$ the sum of the last 2 columns.

So this matrix actually has R^2.

Notes 1) If the columns in A are linearly independent, A is "full rank"
rank = min (M, N)
2) Otherwise A is rank deficient.

Where are we going with this?

The rank of a matrix gives an indication of redundancy in the matrix.

Useful for:

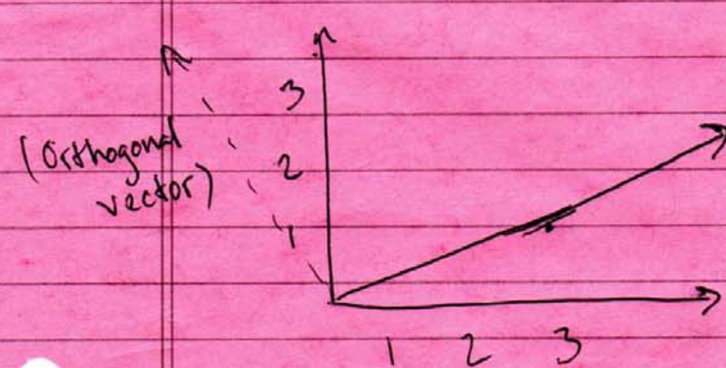
1) Data compression
2) Getting sense of what patterns dominate the matrix.

The directions in R^m that are are not spanned by the columns in A is referred to as the nullspace of A's columns:

Ex

$$A = \begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}$$

The space spanned by the rows in A is R^1 because the space spanned by A's rows lies along (3, 1)



A's rows can describe points along this line only

$$Ax = 0 \rightarrow \underbrace{\begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} -1 \\ 3 \end{bmatrix}}_{x} = 0 \checkmark$$

Eigenvalues and Eigenvectors

The following discussion is based on square matrices.

Given the relationship:

$$Ax_i = \lambda_i x_i$$

A = square matrix
x_i = vector
λ_i = numbers.

- Any λ_i that yields a nontrivial solution to the above is called an eigenvalue.
- The vector x_i corresponding to each λ_i is called an eigenvector of A.

Physically:

- Eigenvectors (x_i) define directions in the the coordinate space along which maximum possible variance can be explained

AND

- In which variance in one direction is orthogonal to the variance explained by other eigen vectors.

Can think of it this way:

$$\begin{array}{cc} N & \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \\ & N \end{array} \begin{bmatrix} e_1 & e_2 & \cdots & e_N \\ \downarrow & \downarrow & & \downarrow \\ & & & \end{bmatrix}$$

A
(N × N)

eigen vectors
(N × N)

$$= \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ & & \ddots & \\ 0 & 0 & \lambda_N \end{bmatrix} \begin{bmatrix} e_1 & e_2 & \cdots & e_N \\ \downarrow & \downarrow & & \downarrow \\ & & & \end{bmatrix}$$

Eigen values
(λ values along diagonal)

Eigenvectors.

When a matrix operates on an eigenvector, only the magnitude changes.

$$(A - \lambda I)\, x = 0 \qquad I = \text{Identity matrix.}$$

This means x lies in the null space of $A - \lambda I$. If A ~~has~~ has eigenvectors, then $A - \lambda I$ must be rank deficient.

If $\det(A) = 0$, then A is singular. Hence choose λ such that $\det(A - \lambda I) = 0$.

<u>Example</u>: Population model:

$$\frac{d\,x(t)}{dt} = a\,x(t) - c\,y(t)$$

$$\frac{dy(t)}{dt} = b\,y(t) - d\,x(t)$$

Growth of x increases with its own population but decreases as y's population increases.

Assume solution of the form

$$x = \alpha e^{\lambda t} \qquad y = \beta e^{\lambda t}$$

(Where λ are eigenvalues)

Substitute into above and divide by $e^{\lambda t}$

$$\begin{bmatrix} a & -c \\ -d & b \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \lambda \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

or $(A - \lambda I)\, x = 0$

$$A - \lambda I = \begin{bmatrix} a & -c \\ -d & b \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a-\lambda & -c \\ -d & b-\lambda \end{bmatrix}$$

The eigenvalues of A are found by solving

$$\det(A - \lambda I) = \lambda^2 - (a+b)\lambda + (ab - cd)$$

Solve this quadratic equation, then substitute in:

$(A - \lambda_i I)\, x_i = 0$ for each root

Solution of this yields eigenvectors x_i

For example

$$A = \begin{bmatrix} 1 & -\frac{3}{2} \\ -\frac{1}{2} & 2 \end{bmatrix}$$

Physically:
1) x will grow exponentially
2) y will grow at $2x$ rate of X.
3) x is more sensitive to competion from y.

$\det(A - \lambda I)$

$$\det \begin{bmatrix} 1-\lambda & -\frac{3}{2} \\ -\frac{1}{2} & 2-\lambda \end{bmatrix} = 0.$$

$$= (1-\lambda)(2-\lambda) - \left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)$$

$$= \lambda^2 - 3\lambda + \frac{5}{4}$$

Solution to $\lambda \rightarrow \lambda = \frac{1}{2}, \frac{5}{2}$

Eigen vectors are found as:

For $\lambda = \frac{1}{2} \rightarrow \frac{1}{2}\begin{bmatrix} 1 & -3 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = 0$

$(A - \lambda I)\, x$

Find that $\alpha = 3\beta$.

For $\lambda = \frac{5}{2} \rightarrow \frac{1}{2}\begin{bmatrix} -3 & -3 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = 0$

Find that $\alpha = -\beta$.

Thus, the eigenvectors of A are:

$\lambda = \frac{1}{2} \rightarrow (3, 1)$ or any multiple thereof

$\lambda = \frac{5}{2} \rightarrow (-1, 1)$ or any multiple thereof.

Solution to original ODE:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{+\frac{1}{2}t}\begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 e^{+\frac{5}{2}t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Cs are determined from initial cond.