Methods for finding coupled patterns

These methods are extensions from basic EOF / PC analysis

* Designed to introduce the concepts, but no HW on these

MCA : Maximum covariance analysis
(also called SVD analysis)
EEOF : extended EOF analysis
CCA : Canonical correlation analysis.

MCA, SVD analysis

Case 1 : SVD of the data matrix $A_{M \times N}$ $\begin{pmatrix} M\text{-time} \\ N\text{-space} \end{pmatrix}$

$$A = U \Sigma V^T$$

The columns in U & V are the eigenvectors of AA^T and A^TA, respectively, or the PCs and EOFs of A.

The diagonal elements in Σ are the singular values. they are the square roots of the eigenvalues of the covariance matrix (C) where $C = A^TA$ or $C = AA^T$.

Case 2: SVD of a dispersion, or covariance, matrix (A^TA or AA^T)

Before $\rightarrow$ $A = U \Sigma V^T$

$C = A^TA$ or AA^T

In this case $\Rightarrow$ $C = V \Sigma^2 V^T$

The columns in V are the orthonormal eigenvectors of A^TA. Note: eigenvectors of C are the same as eigenvectors of C^TC.

The diagonal elements in Σ^2 have units of variance - they are the eigenvalues of A^TA and AA^T

In MCA, apply SVD to the dispersion matrix between 2 different data sets.

$\rightarrow$ datasets have different state space, but a common sampling dimension (e.g. time)

Result of MCA

Patterns in one dataset whose time series explains the largest fraction of variance in the other dataset, or vice versa.

MCA used to isolate coupling between 2 physical fields.

MCA Mathematics

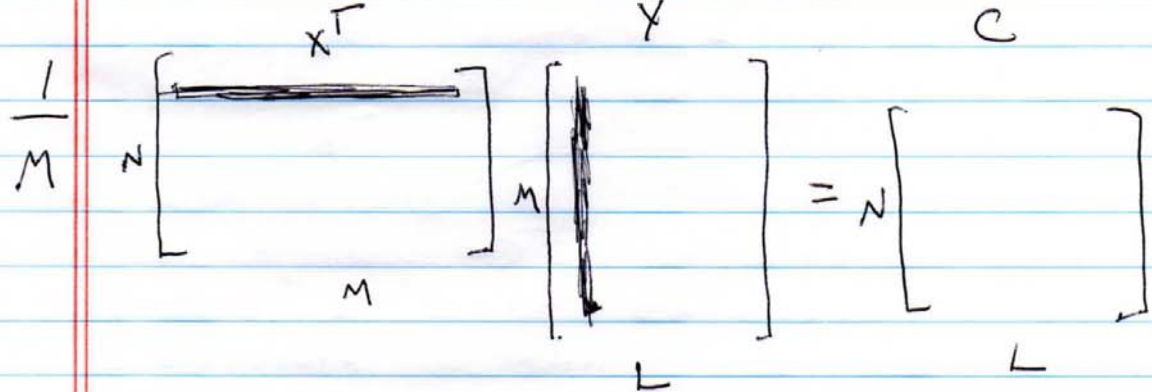
2 data sets : $X_{M \times N}$: left field
$Y_{M \times L}$: right field.

* Note: Slighty different notation in Hartmann.

M = shared sampling dimension (e.g. time)
N & L = structure dimensions.

1) Compute the dispersion matrix (C)

$$C = \frac{1}{M} X^T Y \rightarrow (C_{xy})$$



Dispersion matrix.

Inner dimension M (e.g. time) is consumed by the multiplication.

2) Apply matrix operation SVD to C_{xy}

$$C_{xy} = U \Sigma V^T$$

C_{xy} is $N \times L$ $\quad$ U is $N \times N$ $\quad$ V is $L \times L$

- Columns in $U_{N \times N}$ (referred to as <u>singular vectors</u>) are the column space of C_{xy}. Correspond to structures in dataset X.

- Columns in $V_{L \times L}$ are the row space of C_{xy} and correspond to structures in Y.

- Time series of each pattern in U is found by projecting X onto each pattern in U. Similary, Y is projected on to patterns in V.

- In contrast to EOF analysis, leading singular vector in U corresponds to regression of the left field (X) onto the leading time series in Y, and vice versa.

- Diagonal elements in Σ have units of covariance.

- Expansion coefficient time series are not mutually orthogonal.

Displaying the results

Two types of regression maps can be used:

- Homogeneous regression maps: regress (or correlate) the input data in the left field onto the expansion coefficient time series for the left field. Ditto for right field.

→ Show how singular vectors do in explaining the variance of their own dataset.

- Heterogeneous regression maps: regress (or correlate) ~~the~~ input data in the left field onto the exp. coef. time series for the right field, or vice versa.

→ The resulting maps are characteristic of MCA, as they are the patterns that explain covariance of the 2 data sets.

* ⇒ Usually show heterogeneous regression maps!

If the spatial patterns that explain variance and covariance are similar, the maps should have similar patterns.

Statistical Significance of MCA

Question is similar to problem with EOFs. Basically, does one mode stand out from another?

Fraction of the square covariance explained:

1st mode
$$\frac{\sigma_1^2}{\sum_{i=1}^{N} \sigma_i^2} = \text{Frac. exp.}$$

σ = diagonal elements of Σ

How well coupled are data in general?

Root mean square covariance (RMSC)

the total squared covariance is the square of the diagonal elements in Σ.

$$RMSC = \left(\frac{\sum_{i=1}^{N} \sum_{j=1}^{L} \left(\overline{x_i y_j}\right)^2}{\sum_{i=1}^{N} \overline{x_i^2} \sum_{j=1}^{L} \overline{y_j^2}} \right)^{\frac{1}{2}}$$

→ σ^2's from MCA

→ σ^2's from basic SVD

~ Should be of order 0.1 for well correlated fields.

Other useful numbers: correlation between the leading time series &

Caveats of MCA

- Significance is hard to assess.

- MCA finds coupled patterns, but these do not necessarily reflect physical modes in the data.

- probably best to examine the EOFs of the datasets and their corresponding PCs for strong correlations.

CCA: Canonical Correlation Analysis

Basic idea: perform EOF analysis and MCA in sequence.
- EOF analysis is used to truncate the data sets (X & Y)
- MCA is used to find coupling patterns within the truncated data.

Advantage: Reduces the "noise" in the data, and hence less likely to be impacted by sampling variability.

* Here use Barnett and Preisendorfer (1987) method - see Wilks & von Storch-Zwiers for more details or others.

Treatment of input data

1st step : EOF analysis

- Perform EOF analysis of the original data for both the left and right fields
- Construct the time series of the PCs (EOF amplitudes) at each sampling time.

→ Essentially an orthogonal rotation.

2nd step : Truncation of data.

- Reduce # of degrees of freedom by truncating the data to a small number of EOF/PCs.
- The choice of the number of modes to be retained is tradeoff between explaining as much variance as possible vs. statistical significance. More PCs retained → more variance explained.

3rd step : Normalization

- Normalize each of the retained PCs for left and right fields so the $\hat{PC}$s have unit variance. (i.e. $\sigma_{PC}^2 = 1$)

4^{th} step : Construct correlation matrix

- Using the truncated matrices of normalized PCs for the right and left fields, construct covariance matrix

- Covariance matrix = correlation matrix since PCs have been normalized

5^{th} step :

- Perform MCA on correlation matrix

- Singular values (σ) may be interpreted as correlation coefficients, or "canonical correlations"

- Singular vectors (u and v) correspond to dimensions $N \times N$ and $L \times L$, where N and L are the truncated time values - not the original time value.

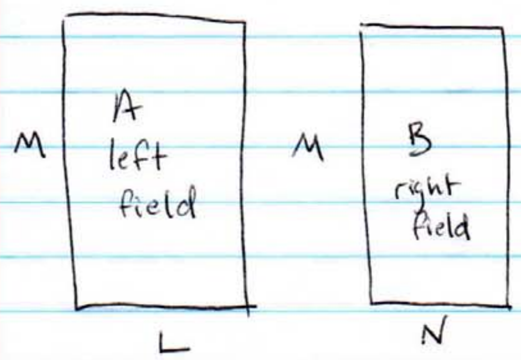
Step 6

- To get canonical correlation vectors, project the matrix of truncated PCs onto the singular vectors.

- Correlate canonical correlation vectors with input data to generate heteorogeneous correlation maps. $\Rightarrow$ Canonical patterns

CCA Schematics

Steps 2/3 - Data truncation, normalization



A & B = input data matrices of truncated PCs (normalized)

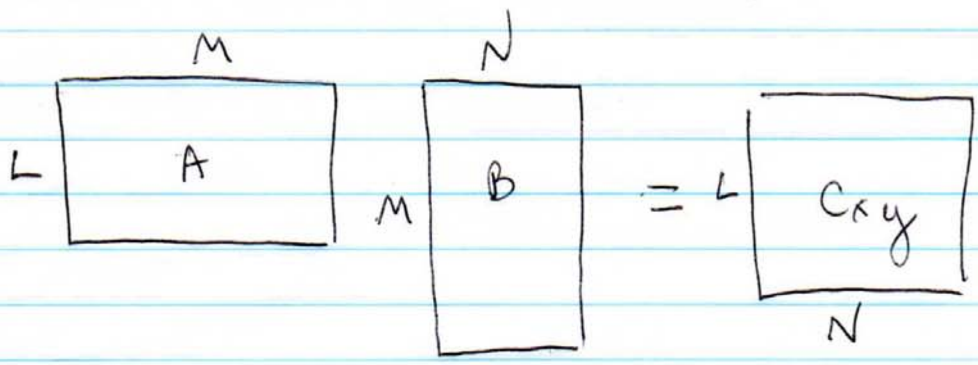
L = # of truncated PCs in A

N = # of truncated PCs in B.

$N < L$

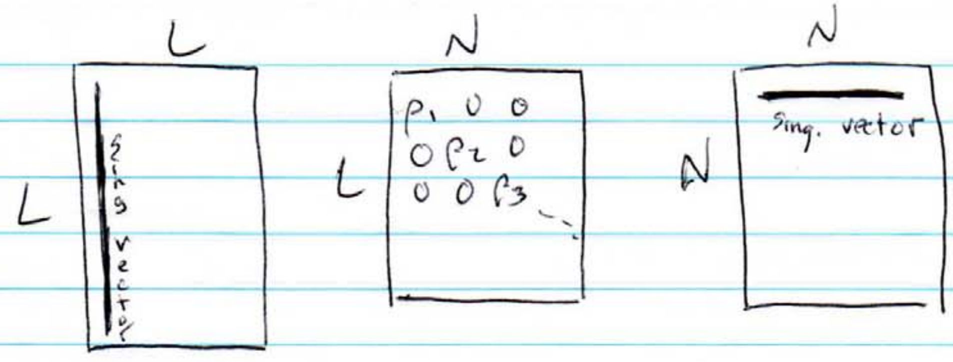
M = time dimension

Step 4 - Construct Correlation matrix



$$\boxed{A^T B = C_{xy}}$$

Step 5 - MCA on correlation Matrix C_{xy}



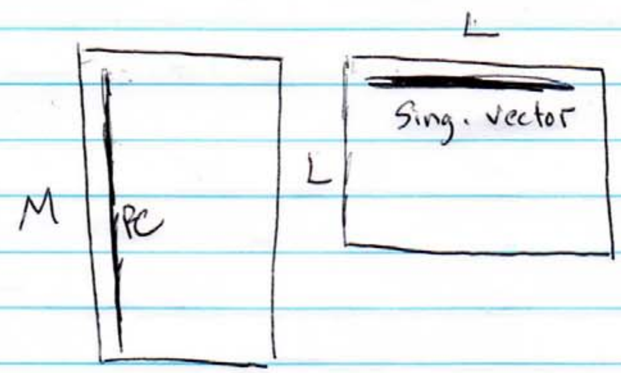
U matrix
Singular vectors of A

ρ matrix
Canonical correlations

V^T matrix
Singular Vectors of B.

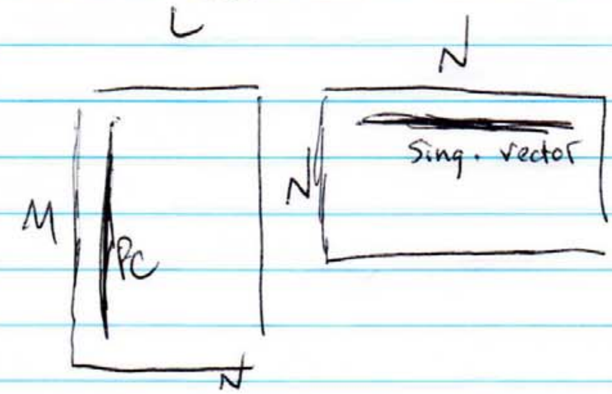
$$\boxed{C_{xy} = U \rho V^T}$$

Step 6 - Project singular vectors onto original data of truncated PCs (A & B matrices) to get canonical correlation vectors.



$\rightarrow AU^T$

(Matrix MxL)



$\rightarrow BV^T$

(Matrix MxN)

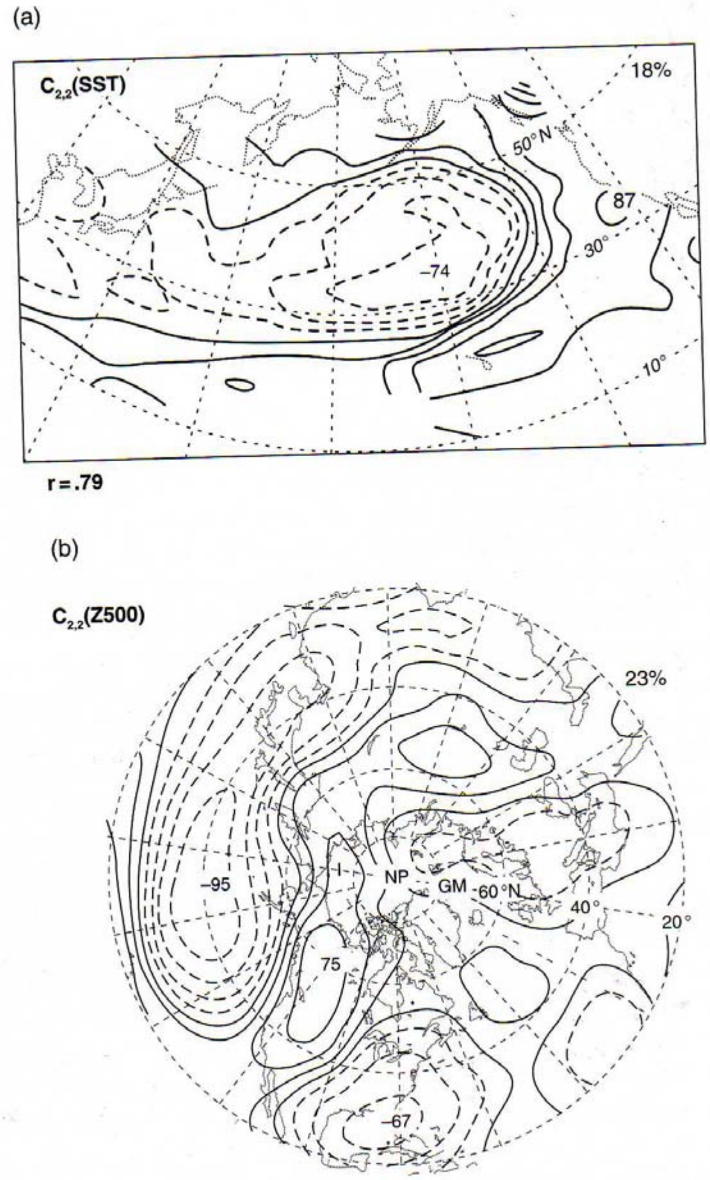


FIGURE 12.1 Homogeneous correlation maps for a pair of canonical variables pertaining to (a) average winter sea-surface temperatures (SSTs) in the northern Pacific Ocean, and (b) hemispheric winter 500 mb heights. The pattern of SST correlation in the left-hand panel (and its negative) are associated with the PNA pattern of 500 mb height correlations shown in the right-hand panel. The canonical correlation for this pair of canonical variables is 0.79. From Wallace *et al.* (1992).