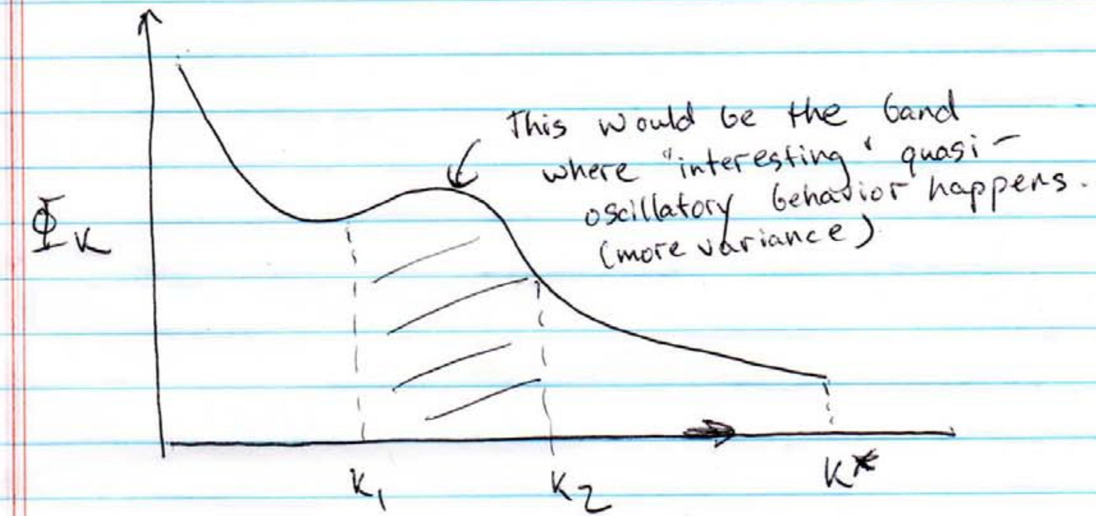
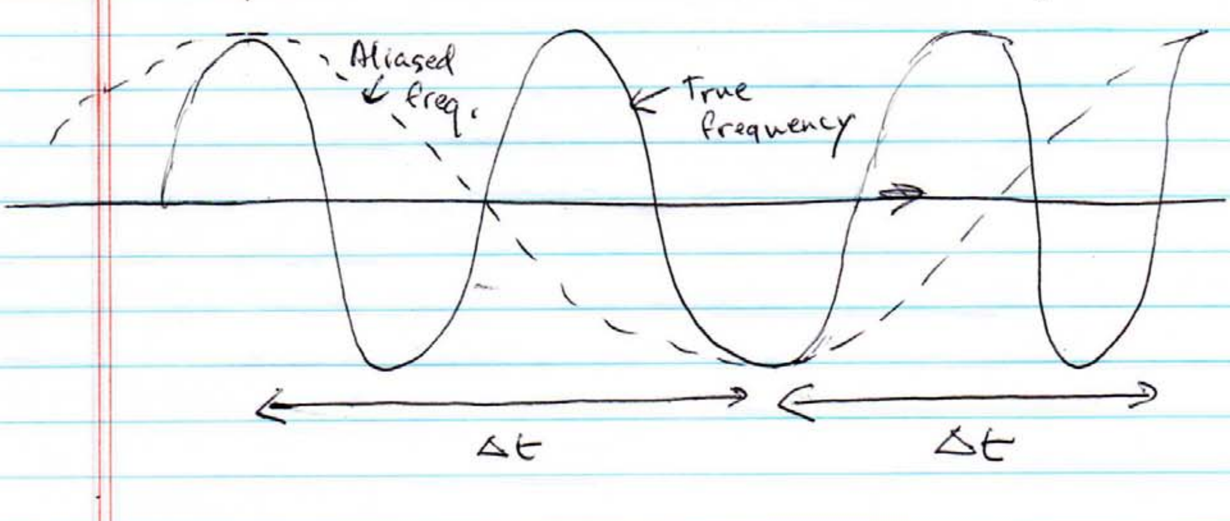
Then for a spectral band:



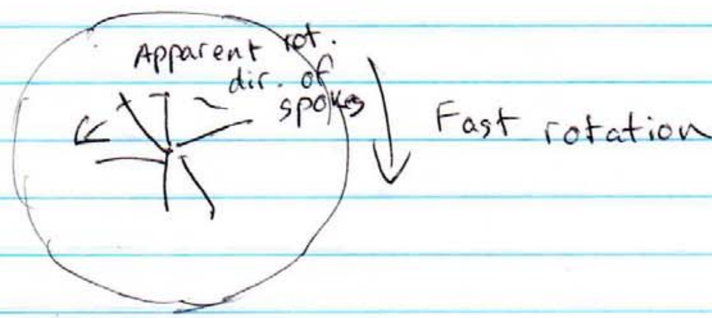
<u>More</u> on the <u>Nyquist</u> <u>frequency</u> & <u>aliasing</u>

Important to know because it is the highest frequency that can be resolved.

If phenomena are occuring at higher frequencies, they will be aliased into lower frequencies



Example of aliasing - a fast rotating wheel, like on a bike or car.



Spokes appear to move against the rotation because the human eye can effectively resolve only so many frames per sec.

Resolution vs. deg. of freedom in estimating $\Phi(k)$

To increase dof in $\Phi(k)$, can do one of 2 things:

1) "Smooth" $\Phi(k)$ by averaging adjacent points in spectrum.
   Tradeoff → ~~loose~~ lose spectral resolution (larger bandwidth)

2) Average $\Phi(k)$ calculated for subsets of the data:
   Tradeoff → lose information about lowest frequencies in the data.

Always choose method that gives quality results → sig. + reproducible.

Adjusted dof

Degrees of freedom for each spectral estimate is $\sim N/m^*$, where m^* is the number of independent spectral estimates, N = no. of data points

As long as the spectrum is tested against red noise (we'll get to a little later) don't need to decrease N to account for autocorrelation.

Methods of computing power spectra

Method #1: Direct Method

Calculate C_k^2 through harmonic analysis (i.e. as per analytic solutions for the discrete Fourier transform).

Easy to code analytic solutions for C_k.

However, algorithm is not efficient, numerous redundancies in calculation of A_k and B_k.

FFT = Fast Fourier Transform
- Exploits redundancies in the 'direct' method
- Available on software packages (e.g. Matlab) and numerical recipes.

Since the Fourier transform assumes cyclic continuity, typically the ends of the time series are "tapered" by data windowing (more on that later)

Remember, spectral estimates calculated via the direct method have very few degrees of freedom (~2 dof)

Ways to get more dof:

1) Average adjacent estimates of Φ (smoothing)

* ..ich way to go depends on how much data you have!!

Hartmann ex: 900 day record

450 spectral est. with 2 dof.
Bandwidth $1/900\ day^{-1}$

Average 10 adjecent estimates
Bandwidth $1/90\ day^{-1} \rightarrow 20$ dof.

2) Average realizations of Φ (break up the time series)

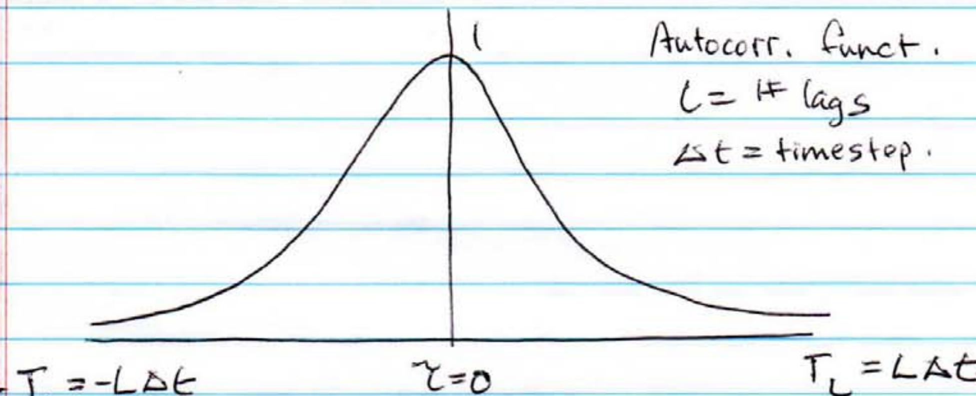
Hartmann ex: 10 series of 900 days.

In this case, if the 10 $\Phi(k)$'s are averaged:
450 spect. est. with 20 dof.
Bandwidth of $1/900\ day^{-1}$

Method 2: 'Lag Correlation' Method.

As per a theorem by N. Wiener, the autocorrelation function and the power spectrum are Fourier transforms of each other.

Hence, power spectrum can be found by performing harmonic analysis on $r(\tau)$ on the interval $-T_L \le \tau \le T_L$ where T_L is the maximum time lag.



Resolution of the spectrum is given by the choice of L.

Spectrum will have $L+1$ spectral estimates with band width of 1 cycle$/2L\Delta t$

Frequencies resolved at: 0, $\frac{1 \text{ cycle}}{2L\Delta t}$, $\frac{2 \text{ cycles}}{2L\Delta t}$, etc.

The unsmoothed $\Phi(0)$ is the mean of the autocorrelation in the interval $-T_L \to T_L$ $\Rightarrow$ variance associated with periods too long to fit in the interval

The values of $\Phi(k)$ for $k \geq 1$ are identified with the cosine coefficients obtained from harmonic analysis expansion of $r(\tau)$ (i.e. A_k's)

Since $r(\tau)$ is an even function about $\tau = 0$, the sine coefficients are all equal to zero (all $B_k = 0$).

As before, dof are gained by
- Smoothing spectrum
- Averaging spectral realizations together.

Red noise spectra

It is shown in the Hartmann notes, that in continuous form, get the following Fourier transform pair relating power spectra and lag autocorrelation.

$$\Phi(\omega) = \int_{-T_L}^{T_L} r(\tau) e^{-i\omega\tau} d\tau$$

$$r(\tau) = \frac{1}{2\pi} \int_{-\omega^*}^{\omega^*} \Phi(\omega)\, e^{i\omega\tau}\, d\omega$$

Can use these relationships to compute power spectrum of red noise:

Autocorrelation function for red noise:

$$r(\tau) = \exp\left(\frac{-\tau}{T}\right) \qquad \begin{array}{l} \tau = \text{lag} \\ T = \text{e-folding timescale.} \end{array}$$

For the power spectrum, then:

$$\Phi(\omega) = \int_{-\infty}^{\infty} \exp\left(\frac{-\tau}{T}\right) \exp^{-i\omega\tau}\, d\tau$$

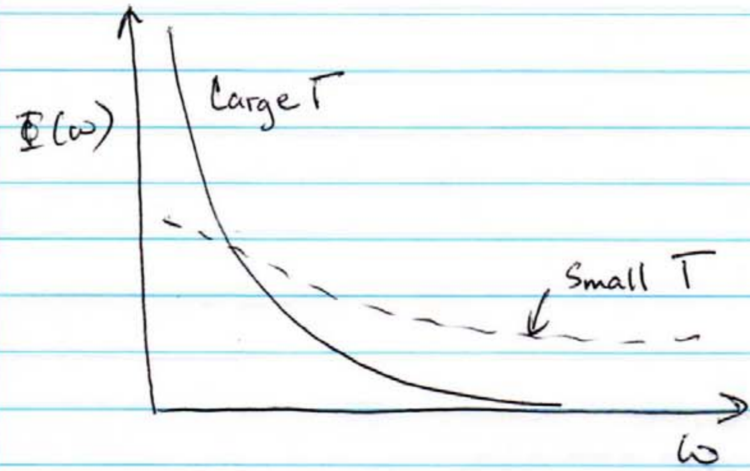
After integration

$$= \frac{-1}{\frac{1}{T} + i\omega} \exp\left\{-\tau\left(\frac{1}{T} + i\omega\right)\right\}_{-\infty}^{\infty}$$

Yields . . .

$$\Phi(\omega) = \frac{2T}{1 + T^2\omega^2}$$

Power spectrum of red noise (graphically)



For white noise, autocorrelation is a delta function of lag

$$r(\tau) = \delta(\tau)$$

$$\Phi(\omega) = \int_{-\infty}^{\infty} \delta(\tau)\, e^{-i\omega\tau}\, d\tau = 1$$

→ Therefore white noise has an equal amount of variance at all frequencies.

Hartmann's notes have idealized plots of different phenomenon and their corresponding spectra.

Plotting the power spectrum

Two basic schemes: linear scale. or log scale.

In the linear scale, plot ω vs. $\Phi(\omega)$ (the power spectral density)

Area under the curve:

$$\int_{\omega_1}^{\omega_2} \Phi(\omega)\, d\omega = \text{Variance explained in } y(t) \text{ between } \omega_1 \text{ and } \omega_2$$

When the frequency interval of interest ranges over several orders of magnitude, it is common to plot $\log(\omega)$ on the abscissa: (natural log)

Linear $\rightarrow \int \Phi(\omega)\, d\omega$

Log $\rightarrow \int \omega\, \Phi(\omega)\, d \ln \omega$

(Hartmann suggests)

In the log plot, $\log(\omega)$ vs $\omega\, \Phi(\omega)$

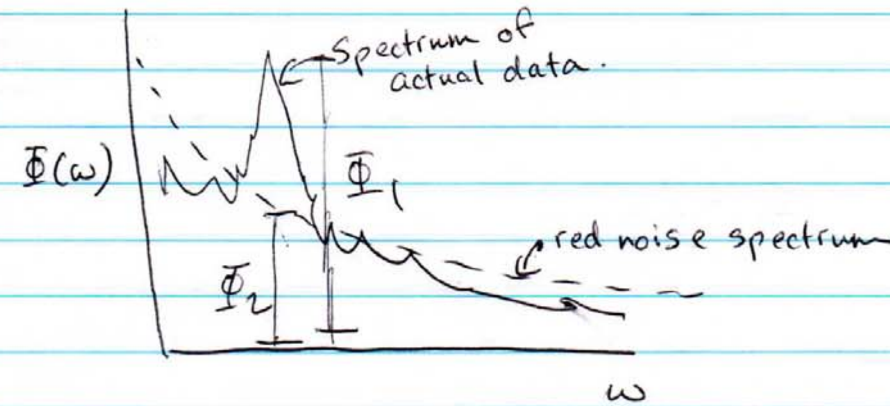
Effectively stretches the low frequency end and contracts high end.

May just plot against $\Phi(\omega)$, as I've seen that done too ...

Statistical significance of spectral peaks

Statistical significance of 'peaks' in the power spectrum is assessed by testing against null hypothesis that the time series is 'noise' ⇒ typically red noise'.

Amplitude of particular peaks is tested against background red noise spectrum.



The significance of the ratio Φ_1/Φ_2 can be assessed using the Chi-squared distribution. ⇒ test Φ_1 against theoretical red noise spectrum.

$$\chi^2 = (n-1)\frac{s^2}{\sigma^2} \qquad \nu = n-1$$

Hartmann's comments on 'a priori' vs. 'a posteriori' testing:

- Apply 'a priori' for χ^2 test if you have a physical reason beforehand to expect a particular phenomena.

- If you did not expect the specific peak, must multiply probability that false (null) hypothesis is true by number of chances to exceed required level. (i.e. indep. spectral estimates)

raise sig. to power m*

For purposes of this class, 'a priori' testing is okay.

Additional comments on statistical sig. of spectral peaks.

The theoretical power spectrum for red noise:

$$\phi(\omega) = \frac{2T}{1+T^2\omega^2}$$

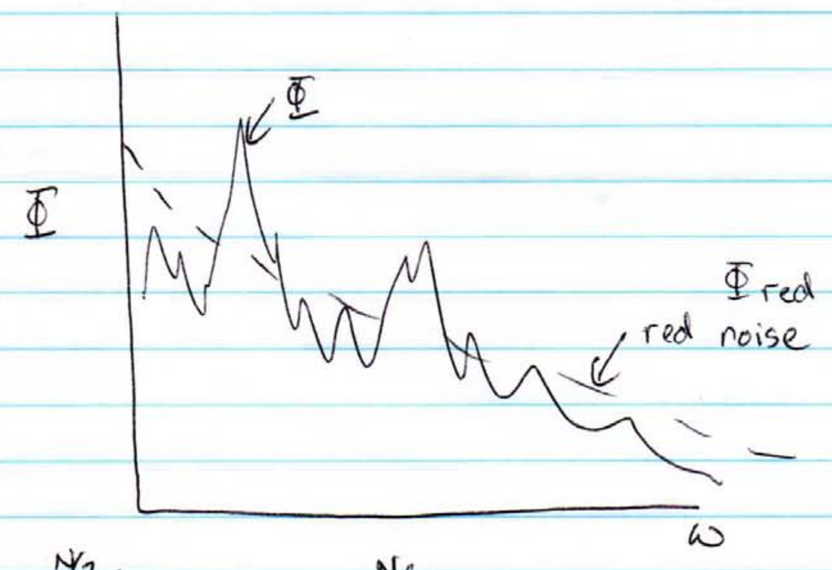
- Applies to infinite data in which all possible lags and frequencies resolved.

In practice (i.e. publications), should use the experimental red noise spectra in Hartmann notes. (e.g. Gilman et al. 1963)

In reality, difference between the experimental and theoretical red noise spectra is very subtle.

For this class, theoretical is okay to use.

You need to make sure that the area underneath the spectrum from acacl data = that of red noise when you plot the spectra!



$$\int_0^{N/2} \Phi(\omega)\,d\omega = \int_0^{N/2} \Phi_{red}(\omega)\,d\omega$$