Data Windows and Window Carpentry

A last, but important topic before we're finally ready to compute spectra.

Basic idea: Have to apply a "windowing" function on the original data before doing spectral analysis

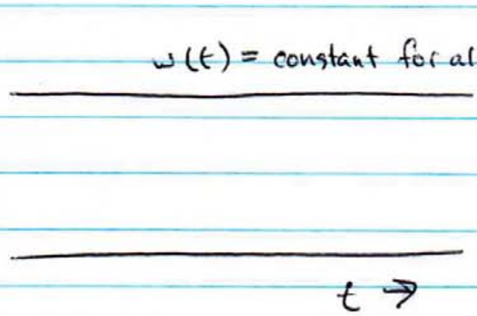
Why is this necessary?

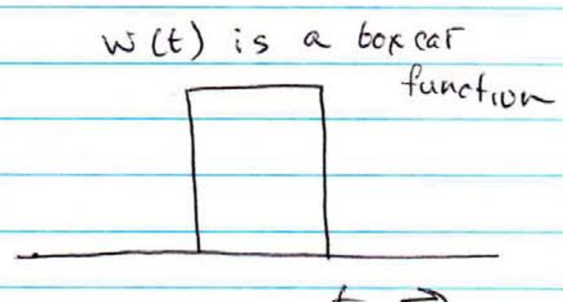
Analytic case: Presumes an infinite domain and the true spectrum can be calculated exactly.

Real world: Cannot observe $f(t)$ (the original data) on the interval $-\infty < t < \infty$, but only through a finite length "window" $w(t)$

* Note $w(t)$ not $\omega(t)$

Infinite domain | Finite domain





Infinite domain

$w(t) = \text{constant (one) for all } t$

So we get an exact representation of $f(t)$, and hence an exact representation of the spectrum.

$$F(\omega) = \int_{-\infty}^{\infty} f(t) w(t) e^{-i\omega t} dt$$

$$= \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

Finite domain

The spectrum is affected by the geometry of the window.

$$F_{adj}(\omega) = \int_{-\infty}^{\infty} f(t) w(t) e^{-i\omega t} dt .$$

Where $w(t)$ is the windowing function.

Using the convolution theorem, can express the above eppression as a combination of the Fourier transform of the data + windowing function.

$$\int_{-\infty}^{\infty} f(t)\, w(t)\, e^{-i\omega t}\, dt = F_{adj}(\omega)$$

If we consider just a single frequency (ω_0), if we integrate around this with respect to ω_0, question is whether get back the true value of $F(\omega)$

$$F_{adj}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, W(\omega - \omega_0)\, d\omega_0 \neq F(\omega)$$

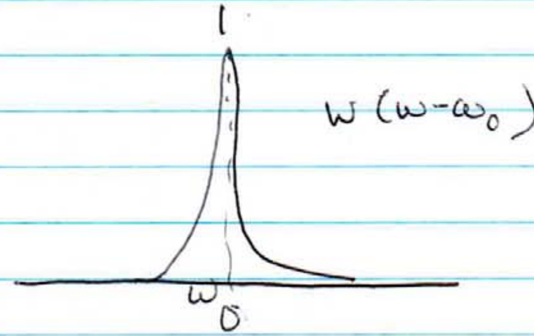
$F(\omega) \rightarrow$ Fourier transform of the data $f(t)$

$W(\omega - \omega_0) \rightarrow$ Fourier trans. of window function $w(t)$ as func. of phase shift.

$W(\omega - \omega_0)$ is called the response function of the window.

<u>Ideal</u> case <u>for $W(\omega - \omega_0)$</u>

Window response function $= 1$ at $\omega = \omega_0$ and zero elsewhere (i.e. delta function at ω_0)



Bartlett Window: Square or rectangular window (Boxcar)

$$w(t) = \begin{cases} \frac{1}{T} & 0 \leq |t| < T \\ 0 & |t| > T \end{cases}$$

where T is the finite domain

(Note: divide through by T to ensure an unbaised spectral estimate).

The Bartlett window is what you're using if:
1) You don't use a window
2) You don't know any better

And it is a common one!

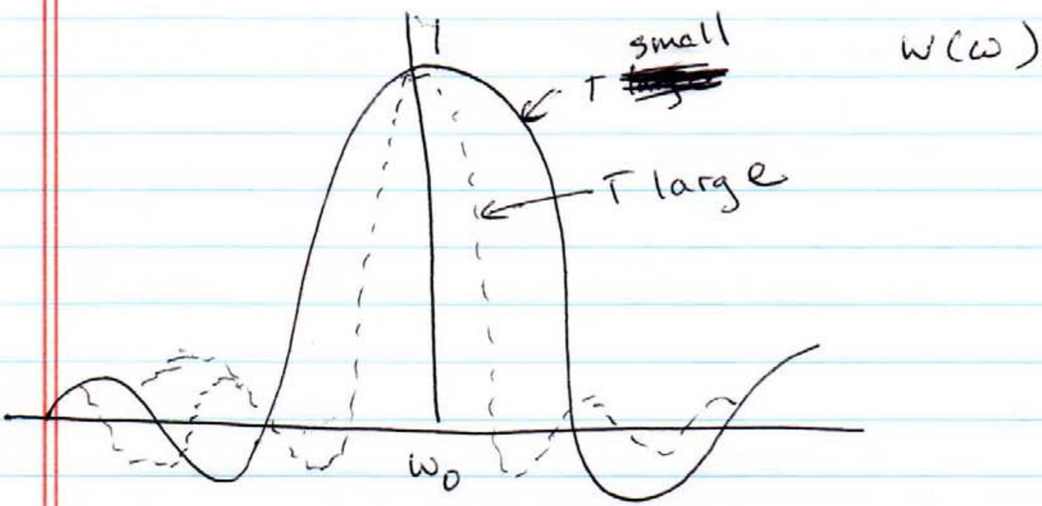
The Fourier transform, or response function, of the boxcar function is:

$$W(\omega) = 2 \,\text{sinc}\left(\frac{\omega T}{2\pi}\right) \quad \text{where} \quad \text{sinc} = \frac{\sin(\pi x)}{\pi x}$$

<u>Effects</u>

- When the spectrum of $f(t)$ is calculated via a Bartlett window, the resulting spectrum is smoothed, but also distorted.

Response function of box car.



Smoothing → from broad hump around ω_0
Distortion → negative side lobes (less than 0)

- The smoothing is generally okay, but the distortion leads to 'spectral leakage'

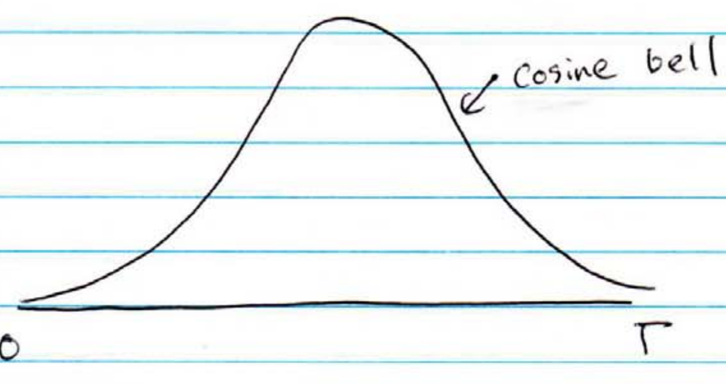
- The impact of the window is a function of T.

The Hanning window (cosine bell taper)

$$w(t) = \frac{1}{T}\left(1 + \cos\frac{2\pi t}{T}\right) \quad \text{for } 0 \le |t| < \frac{T}{2}$$

$$w(t) = 0 \quad \text{for } |t| > \frac{T}{2}$$

Hanning window looks like this:



What the window does:
Acts to taper the ends of the time series.

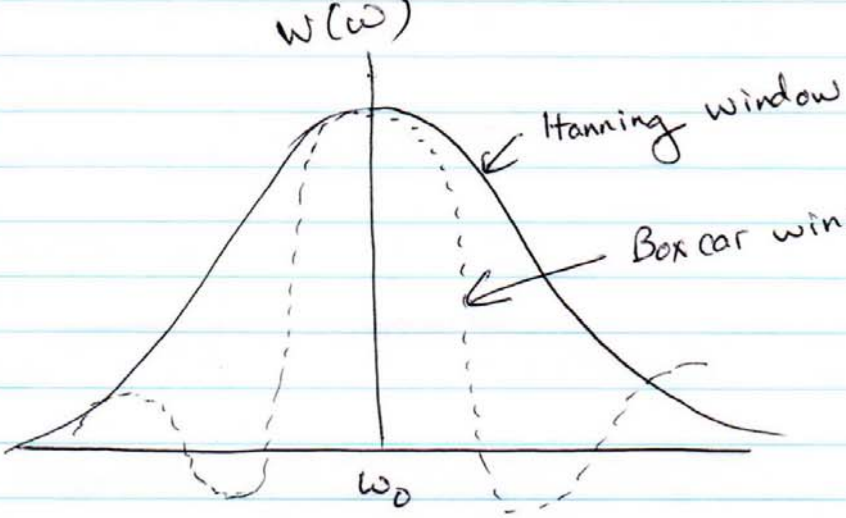
Response function of Hanning window:

$$W(\omega) = \mathrm{sinc}\left(\frac{\omega T}{\pi}\right) + \frac{1}{2}\left[\mathrm{sinc}\left(\frac{\omega T}{\pi}+1\right) + \mathrm{sinc}\left(\frac{\omega T}{\pi}-1\right)\right]$$

Analogous to boxcar function, but 'broader'

Cancels out the negative side lobes.

Comparison of Hanning vs. Box car $W(\omega)$
(for same T)



Advantage: Side lobes are removed, lessening spectral leakage

Disadvantage: Broader response function; so it smooths the spectrum even more.

Other notes on Hanning window:

- Commonly used in meteorology

- Automatically applied in Matlab function spectrum.

Other Windows

Hamming or split cosine bell
- Tapers the last 25% of the data at the ends of the time series
- Provides slightly more optimal reduction in side lobes & more smoothing of central lobe.

Parzen : power window

∴ More information on these in the Hartmann notes + von Storch texts . . .