NOTES AND CORRESPONDENCE

Coherence Significance Levels

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As with many meteorologists and oceanographers, I need to test the statistical significance of coherence between two time series, but have encountered confusion in the literature. Julian (1975) has already discussed this confusion and helped to clear it up. Before finding that reference, I had already made some independent estimates of the significance levels, based solely on the definition of coherence, so the agreement may add further confidence to Julian's (1975) discussion.

If one has n realizations ($k = 1$ to n) of two (real) time-series $x_k(t)$ and $y_k(t)$, one may form their (complex) Fourier coefficients $X_k(\omega)$ and $Y_k(\omega)$, and the *coherence*

$$C(\omega) = \frac{\left| \sum_{k=1}^{n} X_k^*(\omega) Y_k(\omega) \right|}{\sqrt{\sum_{k=1}^{n} |X_k(\omega)|^2 \sum_{k=1}^{n} |Y_k(\omega)|^2}}, \qquad (1)$$

where the asterisk denotes complex conjugate. (Sometimes the summation is done over adjacent frequency bands on an assumption of only slow change with frequency.) If there is no relationship between X and Y, one expects cancellation in the sum, and one expects small coherence as n becomes large. Goodman (1957) gave a formula for significance at size α (e.g., $\alpha = 0.05$):

$$c^2 = 1 - \alpha^{1/(n-1)}. \qquad (2)$$

[To be consistent with this formula, Julian's (1975) text should have used $\alpha = 1 - p$ with $p = 0.95$ in place of the $p = 0.05$ that he used.]

To generate estimates of the significance levels of the coherence, independent of any theory or integrals, a Monte Carlo approach was used: two series of random normals were drawn to make $x(t)$ and $y(t)$ (white noise), using a Fortran library subroutine. These were Fourier transformed to make $X(\omega)$ and $Y(\omega)$, which were substituted into (1) to make C. This was done 1000 times, and the resultant C's sorted. The 90th, 95th and 99th percentiles were picked out. The whole process was repeated 10 times; the mean and standard deviations are entered in Table 1. Each n in Table 1 thus involves 20 000 Fourier transforms. A different seed was used to start the random number generators for each n, so the estimates are independent between number of bands.

The agreement between the Monte Carlo trials and Eq. (2) is quite good, with no systematic differences evident. Using a t-test with nine degrees of freedom, only one difference is significant (for $n = 8$, $\alpha = 0.01$), which is to be expected in a table of this size. Therefore, we can accept (2) as giving the correct significance levels for 4 to 50 bands (8–100 degrees of freedom).

Goodman's (1957) derivation of (2), and the trials here, assumed normally distributed data. A time-series which is strongly non-normal does not satisfy the Wiener-Khintchine theory, and exhibits strong spectral leakage, unless one makes a nonlinear transformation, to *pre-whiten*. Pre-whitening is often recommended as a part of any spectral analysis; the purpose is to bring the actual case closer to the ideal case considered here. However, the normal assumption is quite robust; this is the implication of the Central Limit Theorem, as well as much experience. As an example, the 10 runs of 1000 trials each above for $n = 4$, $\alpha = 0.10$ were (0.736, 0.729, 0.733, 0.738, 0.727, 0.737, 0.747, 0.728, 0.739, 0.728) for an estimate of 0.734 ± 0.006. The call to the normal random number generator was replaced by a call to a uniform random number generator. Now the values in the time-series were uniformly distributed between 0 and 1; not at all normal. The results were (0.725, 0.714, 0.752, 0.737, 0.728, 0.734, 0.743, 0.724, 0.721, 0.741), for an

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TABLE 1. Monte Carlo estimates of the significant coherence and coherence-squared for sizes $\alpha = 0.10$, 0.05, and 0.01 for $n = 4$, 5, 6, 8, 10, 20, 30, 40 and 50 and the prediction from Eq. (2).

	$n = 4$	$n = 5$	$n = 6$	$n = 8$	$n = 10$	$n = 20$	$n = 30$	$n = 40$	$n = 50$
$\alpha = 0.10$									
c	0.734	0.661	0.609	0.537	0.480	0.338	0.275	0.239	0.215
	± 0.006	± 0.004	± 0.009	± 0.007	± 0.006	± 0.005	± 0.005	± 0.004	± 0.002
c^2	0.539	0.437	0.371	0.288	0.230	0.114	0.076	0.057	0.046
(2)	0.536	0.438	0.369	0.280	0.226	0.114	0.076	0.057	0.046
$\alpha = 0.05$									
c	0.793	0.729	0.672	0.595	0.537	0.380	0.314	0.270	0.245
	± 0.008	± 0.004	± 0.009	± 0.009	± 0.010	± 0.005	± 0.005	± 0.005	± 0.004
c^2	0.629	0.531	0.452	0.354	0.288	0.144	0.099	0.073	0.060
(2)	0.632	0.527	0.451	0.348	0.283	0.146	0.098	0.071	0.059
$\alpha = 0.01$									
c	0.887	0.825	0.769	0.704	0.637	0.469	0.388	0.331	0.298
	± 0.009	± 0.013	± 0.011	± 0.004	± 0.012	± 0.012	± 0.008	± 0.007	± 0.008
c^2	0.787	0.681	0.591	0.496	0.406	0.220	0.151	0.110	0.089
(2)	0.785	0.684	0.602	0.482	0.401	0.215	0.147	0.111	0.090

TABLE 2. $c = \sqrt{1 - \alpha^{1/(n-1)}}$ for $\alpha = 0.05$ and $\alpha = 0.01$ for $n = 3$ to 20. Values in boldface refer to the most commonly used values of α and n.

n	$c(0.05)$	$c(0.01)$
3	0.881	0.949
4	0.795	0.886
5	**0.726**	**0.827**
6	**0.671**	**0.776**
7	**0.627**	**0.732**
8	**0.590**	**0.694**
9	**0.559**	**0.662**
10	**0.532**	**0.633**
11	0.509	0.607
12	0.488	0.585
13	0.470	0.565
14	0.454	0.546
15	0.439	0.529
16	0.425	0.514
17	0.413	0.500
18	0.402	0.487
19	0.392	0.475
20	0.382	0.464

estimate of $c = 0.732 \pm 0.011$. The scatter is larger but the agreement is good.

The implication of the trials here is not only that (2) checks out, but that we understand what c, α and n mean. The purpose of this note is to help meteorologists and oceanographers easily test the significance of a coherence. Therefore, Table 2 presents the significance levels for most commonly used values of α and n. I suggest that use of $n < 5$ (10 degrees of freedom) is deceptive, and $n > 10$ (20 degrees of freedom) is wasteful of resolution, under most circumstances.

REFERENCES

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