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Is the Coriolis Force Really Responsible for the Inertial Oscillation?

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Abstract

It is demonstrated that the inertial oscillation is not produced exclusively by "inertial forces," and that the inertial oscillation appears as oscillatory motion even when viewed from a nonrotating frame of reference. The component of true gravity parallel to the geopotential surfaces plays a central role in forcing the inertial oscillation, and in particular it is the only force driving the oscillation in the nonrotating reference frame.

1. Introduction

Horizontal momentum equations describing inviscid midlatitude motions may be written in the approximate form,

$$\frac{du}{dt} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \qquad (1)$$

$$\frac{dv}{dt} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \qquad (2)$$

where p is pressure, ρ is density, and u and v are the westerly and southerly wind velocities in a Cartesian coordinate system lying in a plane tangent to the earth's geopotential surfaces with the x axis oriented east–west (Holton 1992, p. 40). As in the standard "f-plane" approximation, suppose that the Coriolis parameter f is constant. A classical exercise in basic atmospheric dynamics is to neglect vertical motions and examine those solutions to (1) and (2) for which the wind speed is constant. One such solution, the *inertial oscillation,* is obtained if the pressure gradients in (1) and (2) are identically zero, in which case

$$\frac{du}{dt} - fv = 0, \qquad (3)$$

$$\frac{dv}{dt} + fu = 0. \qquad (4)$$

Differentiating (3) with respect to time and using (4) to eliminate dv/dt yields

$$\frac{d^2u}{dt^2} + f^2u = 0,$$

which has sinusoidal solutions of period $2\pi|f|^{-1}$. If $u = u_0$ and $v = 0$ at some initial time $t = 0$, the velocity at later times is

$$u(t) = u_0\cos ft, \quad v(t) = -u_0\sin ft.$$

An air parcel initially located at the origin follows the *inertia circle* trajectory

$$x(t) = \frac{u_0}{f}\sin ft, \quad y(t) = \frac{u_0}{f}(\cos ft - 1). \qquad (5)$$

The radius of curvature of this trajectory is $u_0 f^{-1}$ and the direction of travel is anticyclonic.

Inasmuch as the Coriolis force is the only force in equations (3) and (4), it is tempting to assume that the Coriolis force is responsible for the inertial oscillation. Indeed the name "inertial oscillation" suggests that the motion arises solely as a result of *inertial forces,* which are apparent forces that appear in accelerating coordinate frames (e.g., Coriolis and centrifugal forces). The *Glossary of Meteorology* (Huschke 1959) reinforces this idea by defining "inertial flow" as "flow in the absence of external forces." In fact, inertial oscillations are not produced entirely by inertial forces, and even when viewed from a nonrotating coordinate frame, the inertial oscillation looks like oscillatory motion. One external force plays an essential role in driving the inertial oscillation, and that force is gravity.

2. The inertial oscillation as viewed from a nonrotating reference frame

In order to better appreciate the role of gravity in the inertial oscillation, it is helpful to describe the oscillation that would be seen by an observer in a nonrotating coordinate frame. Consider, therefore, an f plane tangent to the earth at the North Pole. Defining horizontal and angular velocity vectors as

$$\mathbf{V_r} = u\mathbf{i} + v\mathbf{j}, \quad \mathbf{\Omega} = \frac{f}{2}\mathbf{k},$$

(3) and (4) may be written as the single vector equation

$$\frac{d\mathbf{V_r}}{dt} + 2\mathbf{\Omega} \times \mathbf{V_r} = 0. \tag{6}$$

The acceleration in a reference frame rotating at angular velocity $\Omega = |\mathbf{\Omega}|$ is related to the acceleration in an fixed reference frame via the formula

$$\frac{d\mathbf{V_f}}{dt} = \frac{d\mathbf{V_r}}{dt} + 2\mathbf{\Omega} \times \mathbf{V_r} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{R}), \tag{7}$$

where $\mathbf{V_f}$ is the horizontal velocity vector in the fixed reference frame and **R** is a vector originating at the axis of rotation and terminating at the instantaneous position of the air parcel (Holton 1992, p. 32). Substituting (7) in (6), the equation governing the inertial oscillation in the fixed coordinate frame becomes

$$\frac{d\mathbf{V_f}}{dt} = \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{R}). \tag{8}$$

The right-hand side of this equation must represent a real external force since (8) describes motion in a nonrotating coordinate frame. This force is the component of true gravity directed parallel to the polar f plane. As indicated in Fig. 1, the net force on each point on the earth's surface, the "apparent gravity," is the vector sum of true gravity and the centrifugal force due to the earth's rotation about its polar axis. Except at the equator and the poles, the apparent gravity vector is directed equatorward of the true gravity vector. Since the earth's crust cannot support a shearing stress, the earth assumes the shape of an oblate spheroid whose surface is everywhere normal to the apparent gravity vector. One consequence of the resulting equatorial bulge in the earth's geopotential surfaces is that there exists a poleward component of true gravity parallel to the geopotential surfaces at all latitudes except 0° and 90°; or more simply, the equator is uphill.

Objects located at a fixed point on the surface of the rotating earth do not "fall" poleward because the poleward component of true gravity is exactly balanced by an equatorward component of the centrifugal force. Consider, however, the behavior of a moving air parcel whose instantaneous position is the North Pole. This parcel is not rotating around the earth's polar axis, and unless some torque acts to change its angular momentum about the polar axis, it will never rotate with the underlying earth and never be subject to an equatorward centrifugal force. The motion of this parcel can be easily modeled with the polar f-plane equations. Let us first determine the parcel motion with respect to a nonrotating coordinate system. If the origin coincides with the North Pole, (8) may be expressed in component form as

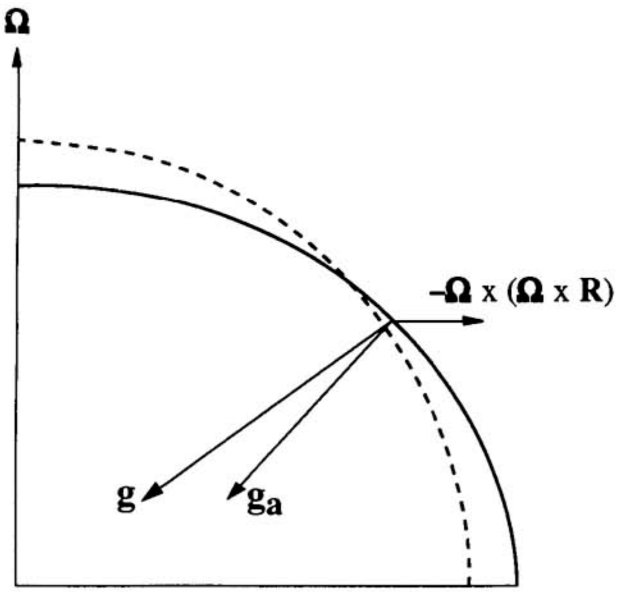

FIG. 1. Deformation of the earth's surface from a true sphere (dashed line) to an oblate spheroid (solid line). True gravity, indicated by the vector g, is perpendicular to the spherical surface. Apparent gravity g_a, being the vector sum of g and the centrifugal force $-\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{R})$, is perpendicular to the surface of the oblate spheroid.

$$\frac{d^2x}{dt^2} + \Omega^2 x = 0, \quad \frac{d^2y}{dt^2} + \Omega^2 y = 0. \tag{9}$$

Recalling that $\Omega = f/2$, the fixed-coordinate trajectory for an air parcel leaving the pole at $t = 0$ with initial velocities $u = u_0$, $v = 0$ is

$$x = \frac{2u_0}{f}\sin\left(\frac{ft}{2}\right), \quad y = 0. \tag{10}$$

This trajectory is a straight line segment along which the parcel oscillates with period $4\pi f^{-1}$. In contrast, the trajectory of the same parcel, as viewed in a coordinate system rotating with the earth, is the familiar

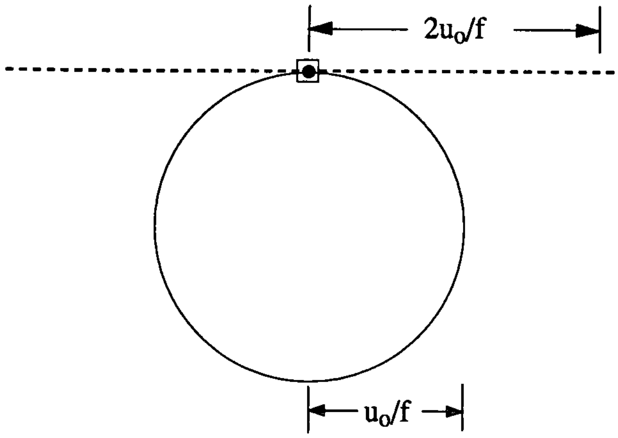

FIG. 2. Fixed-coordinate trajectory [dashed, Eq. (10)] and rotating-coordinate trajectory [solid, Eq. (5)] for an air parcel leaving the pole at $t = 0$ with initial velocities $u = u_0$, $v = 0$. The initial position of the parcel is indicated by the open square. The North Pole is shown as a heavy dot.

inertia circle (5). These fixed- and rotating-coordinate-frame trajectories are compared in Fig. 2. Note that the length of the fixed-frame trajectory is twice the diameter of the inertia circle and the frequency of the fixed-frame oscillation is one-half that observed in the rotating frame.

The relative orientation of the fixed- and rotating-frame trajectories shown in Fig. 2 changes as the earth rotates because the inertia circle travels around the pole with the rotating earth. Figure 3 shows snapshots of the relative position of the two trajectories at 3, 6, 9, and 12 h. In all four panels, the air parcel occupies a point indicated by the open square at the intersection of the fixed- and rotating-frame trajectories. At $t = 0$ (Fig. 2), the air parcel is at the North Pole moving toward the right along both trajectories. Three hours later (Fig. 3a), the inertia circle has rotated 45° counterclockwise around the pole, and the air parcel lies at the intersection of the inertia circle and the fixed-frame trajectory at a distance $u_0\,[(2)^{1/2}f]^{-1}$ to the right of the pole. At six hours (Fig. 3b), the air parcel has just reached the end of its rightward trajectory in the fixed coordinate frame. This position coincides with the most equatorward point on the inertia circle trajectory, which has now rotated 90° counterclockwise from its initial position. At this time, the parcel has completed one-half its rotating-frame orbit and one-quarter of its fixed-frame oscillation. At 12 h (Fig. 3d), the air parcel has returned to the pole, completing one orbit around the inertia circle, but only the right half of its fixed-frame linear oscillation. Note that although in Figs. 3a, 3b, and 3c the y component of the air parcel's earth-relative velocity is nonzero, the y-component velocity is zero in the fixed coordinate frame because the ground-relative motion of the parcel is exactly compensated by the movement of the underlying earth.

As indicated by (10) and Fig. 3, the inertial oscillation of an air parcel passing through the pole appears as straight-line simple harmonic motion when viewed in a nonrotating coordinate frame. If a parcel does not pass through the pole but remains within the polar region so that its motion can still be well approximated using the polar f plane, its nonrotating coordinate frame trajectory will be given by the general solution to (9),

$$x = A\cos\Omega t + B\sin\Omega t, \qquad y = C\cos\Omega t + D\sin\Omega t.$$

Suppose the initial position of the air parcel is some distance r_0 from the pole and the parcel is given arbitrary initial *earth-relative* velocities u_0, v_0. Without loss of generality, the x axis in the fixed-coordinate system can be oriented to pass through the initial position of the parcel, in which case the initial conditions for motion in the fixed-reference frame are

$$x(0) = r_0,\ \ y(0) = 0,\ \ u_f(0) = u_0,\ \ v_f(0) = v_0 + \Omega r_0.$$

The specific solution to (9) satisfying these initial conditions is

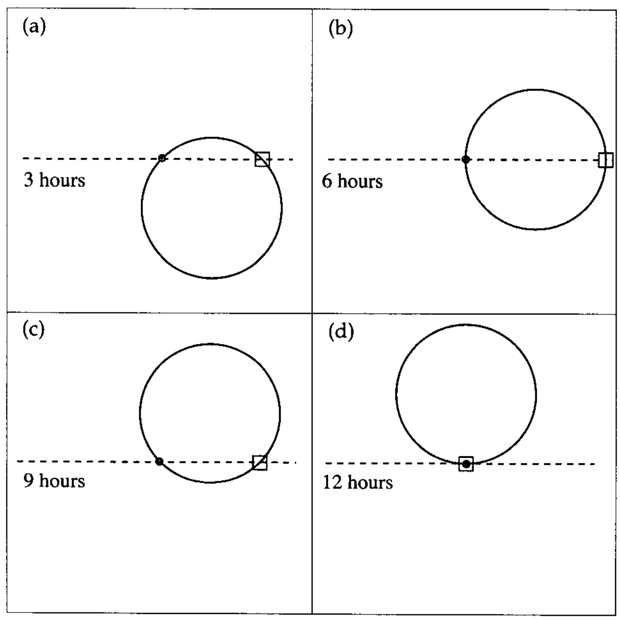

FIG. 3. Relative position of the fixed- and rotating-coordinate trajectories shown in Fig. 2 at: 3, 6, 9, and 12 h. As in Fig. 2 the position of the air parcel is indicated by the open square. The North Pole is indicated by the heavy dot. The inertia circle trajectory rotates with the earth, turning 180° around the pole in 12 h.

$$x(t) = r_0 \cos \Omega t + \frac{u_0}{\Omega} \sin \Omega t, \qquad (11)$$

$$y(t) = r_0 \sin \Omega t + \frac{v_0}{\Omega} \sin \Omega t. \qquad (12)$$

The trajectory defined by the preceding is an ellipse. It is, however, more instructive to consider the total trajectory as the sum of two types of motion. The first term in each of these equations represents the circular orbit followed by the initial position of the parcel as the earth rotates about its axis. The second terms in (11) and (12) have the same form as (10) and, as before, these terms represent gravitationally driven simple harmonic oscillations along a line segment parallel to the initial velocity vector. If one were to represent the same trajectory in coordinates rotating with the earth, the first terms in (11) and (12) would disappear, and the remaining terms would deform from line segments to the familiar inertia circle trajectories.

If an air parcel is sufficiently far from the pole that the polar f-plane approximation is inadequate, its fixed-frame trajectory is not confined to a two-dimensional plane and the motion is more difficult to calculate. [See Stommel and Moore (1989) for a detailed discussion of inertial trajectories on the rotating sphere.] It is nevertheless easy to demonstrate the influence of real (i.e., nonapparent) forces on the inertial oscillation. Suppose an air parcel at latitude ϕ is initially moving north and undergoing an inertial oscillation. To within the accuracy of the midlatitude f-plane approximation, the parcel moves around the inertia circle shown in Fig. 4 with a period relative to the rotating earth of

$$\frac{\pi}{\Omega \sin \phi}. \qquad (13)$$

An observer in a fixed reference frame can obtain some rudimentary information about the parcel's motion by sampling its position at one-day intervals, over which time the f plane completes one full revolution about the earth's axis and returns to its original position with respect to a fixed frame of reference.[1] At the instant when the f plane returns to its initial position, the air parcel must lie at some point on original inertia circle trajectory shown in Fig. 4. The fraction of the total inertial circle orbit traversed by the parcel during the one-day sampling interval is 2 sinϕ, which is the ratio of the period of the earth's rotation to the period of the inertial oscillation. According to (14), the air parcel could reappear at any position on the inertia circle, depending on its initial latitude ϕ. A typical example is shown in Fig. 4, which could correspond to either 11° or 43°N latitude. The velocity of the parcel shown in Fig. 4 has clearly changed over the 24-h period between day 1 and day 2. Since this change of velocity is observed in a fixed frame of reference, it must be produced by noninertial forces. As in the preceding examples, gravity is the external force responsible for the observed acceleration.

[1]Neglecting the motion of the earth around the sun and other astronomical movements.

3. Conclusions

It has been demonstrated that the inertial oscillation is not produced exclusively by "inertial forces," and that the inertial oscillation appears as oscillatory motion even when viewed from a nonrotating frame of reference. Although not explicitly included in the f-plane equations (3) and (4), the component of gravity parallel to the geopotential surfaces plays a central role in forcing the inertial oscillation and in particular, it is the only force driving the oscillation in the fixed reference frame. Since noninertial forces play a crucial role in the "inertial oscillation," it might be preferable to rename the phenomenon. Gravitational forces prevent air parcels undergoing inertial oscillations from conserving linear momentum, but gravity does not exert a torque about the earth's polar axis and, as a consequence, air parcels undergoing inertial oscillations conserve angular momentum about the polar axis. Thus, a possible new name for the inertial oscillation might be the "constant angular momentum oscillation." The "inertia circle" might be better described as a "constant angular momentum orbit."

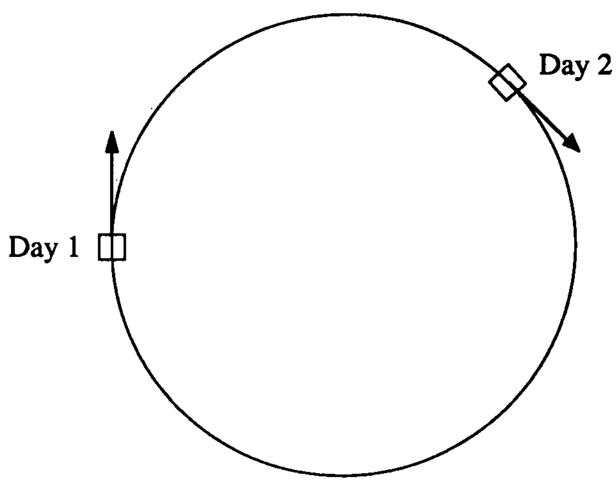

FIG. 4. Initial position and velocity of an air parcel on an inertial trajectory (day 1). Subsequent position and velocity 24 h later (day 2).

Much of the confusion associated with the interpretation of the f-plane equations arises because they are assumed to describe motion in a horizontal plane rotating about a vertical axis. As discussed in connection with (7) and (8), the centrifugal forces that should appear in the rotating-frame equations governing such motion are not present in (3) and (4) because they are exactly cancelled by gravity. A correct physical model for the f plane needs to account for the presence of the gravitational restoring force that balances these centrifugal forces. One relatively easy way to provide the necessary restoring force is to deform the rotating plane into a rotating paraboloidal dish. Inertial motion within a paraboloidal dish, and within other surfaces of revolution, is discussed in considerable detail by Stommel and Moore (1989). An alternative derivation of the paraboloidal dish model, which differs from that in Stommel and Moore, is presented in the Appendix.

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Appendix: Building a physical model for the f plane

A physical model for the inertial motion an air parcel on an f plane can be constructed by placing a hockey puck in a circular bowl of ice that opens upward and rotates around a vertical axis perpendicular to the bottom of the bowl at angular velocity Ω. If the height of the bowl is $h(x, y)$, an equation for the three-dimensional surface defining the bowl may be written in the form

$$S(x, y, z) = 0, \text{ where } S(x, y, z) = h(x, y) - z. \quad \text{(A1)}$$

If surface friction is neglected, the only forces acting on a hockey puck within the bowl will be gravity and the normal force exerted on the puck by the surface of the bowl. The normal force per unit mass can be expressed as $N\nabla\mathbf{S}$, where $\nabla\mathbf{S}$ is a vector normal to the surface of the bowl and N is a scalar determining the magnitude of the normal force. Newton's law governing the puck's accelerations in a fixed reference frame may thus be written as

$$\frac{d\mathbf{V_f}}{dt} = -g\mathbf{k} + N\nabla\mathbf{S}, \quad \text{(A2)}$$

where $\mathbf{V_f}$ is the three-dimensional vector of velocities with respect to the nonrotating coordinate frame. Let the x, y, and z components of $\mathbf{V_f}$ be denoted as u_f, v_f, and w_f, respectively. A closed system of four scalar equations in the four unknowns u_f, v_f, w_f, and N is formed by (A2) and the kinematic condition that the velocity of the puck perpendicular to the bowl must be zero

$$\mathbf{V_f} \bullet \nabla\mathbf{S} = 0. \quad \text{(A3)}$$

An explicit equation for N can be obtained by taking the dot product of $\nabla\mathbf{S}$ with the equations of motion (A2) and using the relation

$$\nabla\mathbf{S} \bullet \frac{d\mathbf{V_f}}{dt} = \frac{d(\nabla\mathbf{S} \bullet \mathbf{V_f})}{dt} - \mathbf{V_f} \bullet \frac{d\nabla\mathbf{S}}{dt} = -\mathbf{V_f} \bullet \frac{d\nabla\mathbf{S}}{dt}, \quad \text{(A4)}$$

to yield

$$N = -\frac{g + \mathbf{V_f} \bullet d\nabla\mathbf{S}/dt}{|\nabla\mathbf{S}|^2}. \quad \text{(A5)}$$

If the rotating ice bowl is to serve as a model for inertial motion on the f plane, its shape should be chosen so that the equations governing the motion of the hockey puck are identical to those describing air parcel motion in the absence of pressure gradients. The equations governing these two systems can be most easily compared by transforming (A2) into a coordinate frame in which the x and y axes are rotating at the same angular velocity as the ice bowl, the z axis coincides with the axis of rotation of the bowl, and the origin is at the bottom of the bowl. Define

$$\mathbf{V_r} = u_r\mathbf{i} + v_r\mathbf{j} + w_r\mathbf{k} \quad \text{(A6)}$$

as the velocity vector with respect to the rotating coordinates, $\mathbf{\Omega} = \Omega\mathbf{k}$ as the angular velocity vector, and $\mathbf{R}$ as a vector originating at the axis of rotation and terminating at the instantaneous position of the hockey puck. Substituting (7) into (A2) yields the following scalar equations for the velocity components in the rotating coordinate frame

$$\frac{du_r}{dt} - 2\Omega v_r - \Omega^2 x = N\frac{\partial h}{\partial x}, \quad \text{(A7)}$$

$$\frac{dv_r}{dt} + 2\Omega u_r - \Omega^2 y = N\frac{\partial h}{\partial y}, \quad \text{(A8)}$$

$$\frac{dw_r}{dt} = -g - N. \quad \text{(A9)}$$

If vertical accelerations are neglected, which is akin to the hydrostatic approximation, (A9) reduces to $N = -g$. Under this approximation, (A7) and (A8) will be equivalent to the f-plane system (3) and (4), when

$$\frac{f}{2} = \Omega, \quad \Omega^2 x = g\frac{\partial h}{\partial x}, \quad \text{and} \quad \Omega^2 y = g\frac{\partial h}{\partial y}. \tag{A10}$$

The first requirement is completely natural since $f/2$, being the angular velocity about an axis perpendicular to the x–y plane in the f-plane approximation, is directly analogous to Ω in the ice bowl system. The second two requirements are satisfied if the bowl is shaped like the circular paraboloid

$$h(x,y) = \frac{\Omega^2}{2g}\left(x^2 + y^2\right). \tag{A11}$$

Thus, to within the accuracy of the "hydrostatic" approximation $N = -g$, the motion of a hockey puck within the paraboloidal ice dish (A11) will be identical to the inertial motion of an air parcel on the f plane.

The accuracy of the "hydrostatic" approximation can be assessed by examining the exact expression for N, which may be evaluated from (A5) using (A11) and the definition of S to yield

$$N = -g\left[\frac{1+\left(u_f^2 + v_f^2\right)\Omega^2 g^{-2}}{1+\left(x^2 + y^2\right)\Omega^4 g^{-2}}\right]. \tag{A12}$$

According to this equation, N will be well approximated by $-g$, provided

$$g^2 >> \Omega^2(u_f^2 + v_f^2) \tag{A13}$$

and

$$g^2 >> \Omega^4(x^2 + y^2). \tag{A14}$$

The inequality (A13) reduces to (A14) under the assumption that the total fixed-frame velocity scales like the maximum straight-line velocity associated with the rotation of the ice bowl—that is, like the maximum of $\Omega(x^2 + y^2)^{1/2}$. If experiments were conducted inside a bowl rotating at 10 rpm, (A14) would be satisfied, and the "hydrostatic" approximation would be valid, provided the puck remained within 4 m of the axis of rotation.

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