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Potential vorticity and the electrostatics analogy: Ertel–Rossby formulation

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SUMMARY

The isomorphism between the theory of electrostatics and the quasi-geostrophic potential vorticity is extended to the Ertel–Rossby potential vorticity. Anomalies of mass-weighted potential vorticity are defined relative to an arbitrary zonal-mean or horizontal-average flow and given in terms of the divergence of a vector field. The vector is the sum of linear and non-linear contributions and can be written as a dielectric tensor acting on the geopotential gradient. The linear components of the tensor differ from those for the quasi-geostrophic potential vorticity only if there exists a vertical variation of background potential vorticity, such as occurs at the tropopause, or if there is shear of the assumed background flow. The non-linear components are absent in the quasi-geostrophic case.

The forms of free, bound and total charge are defined for accurate non-linear forms of the potential vorticity. The free-space Green's function for the operator defining the total charge is identical to that for quasi-geostrophic theory and provides a scheme whereby the field attributed to each potential vorticity element is an invariant quantity. One of the most important results arising from this formulation is that the non-linearities in the definition of potential vorticity can be neglected when considering the far-field effect of potential vorticity anomalies. An analytical example of these ideas is given for a uniform anomaly of semi-geostrophic potential vorticity embedded in an otherwise uniform background potential vorticity. The dielectric constant and bound charge are calculated and give a clear insight into the differences between this and the quasi-geostrophic solution.

KEYWORDS: Electrostatics analogy Potential vorticity

1. INTRODUCTION

In Bishop and Thorpe (1994) (hereafter BT) the isomorphism between a quasi-geostrophic potential vorticity anomaly and a static electrical charge is described. This gives a convenient picture of how potential vorticity (PV) anomalies are associated with streamfunction fields. In this paper we consider the extension of these ideas to the Ertel–Rossby form of the PV. The PV can be written in a form proportional to the divergence of a vector field: $\mathrm{PV} = 1/\rho\nabla \cdot (\boldsymbol{\zeta}\theta)$. The vector field is the product of the three-dimensional vorticity vector, $\boldsymbol{\zeta}$, and the potential temperature, θ. The divergence form of the Ertel–Rossby PV is fundamental because it shows that the mass-weighted volume integral of PV only depends on the normal component of the vector field at the boundaries. This property is a familiar one and is shared by the quasi-geostrophic form and electrical charges and their fields.

There is, in principle, a substantial difference between the quasi-geostrophic form (q) and the full form (PV) for the potential vorticity. The former is a linear quantity which can be interpreted as a modification of the vertical component of absolute vorticity. On the other hand the PV is a quadratic quantity with contributions from the product of vertical component of vorticity and static stability. One might conclude therefore that there was no electrostatics analogy for PV but this proves not to be the case. The electrostatics analogy described here helps to bridge the conceptual gap that exists in interpreting effects due to anomalies of quasi-geostrophic and Ertel–Rossby PV.

Here it will be shown that the field due to an Ertel–Rossby PV anomaly is composed of a linear and a non-linear component. This non-linear part is large only close to the anomaly and is interpretable in terms of bound charge, as introduced by BT. It is known in quantum mechanics, Bjorken and Drell (1964), that very close to an electric charge Coulomb's law

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is modified in a non-linear way to include the effect of a so-called vacuum polarization. This is a quantum effect best visualized in terms of the seemingly contradictory notion of a polarizability of the vacuum. Hence there is a conceptual similarity between the two systems.

The motivation for this research is to cast light on the problem of attribution where the flow and temperature associated with particular PV anomalies are sought. This is required within the framework of piecewise PV inversion as discussed by Davis and Emanuel (1991). The insight provided by the analogy with electrostatics gives a framework based on physical reductionist principles to interpret the building blocks of atmospheric dynamics, namely PV anomalies. This analogy relies heavily on the field theoretical approach of electrostatics. The non-linearity of the field, although a complication, does not imply that the attribution concept is flawed. As discussed by BT and by Davis and Emanuel (1991) the role of the Earth's surface is also a complication but one which is present in analogous electrostatic systems such as a charge adjacent to a conducting sheet. Further discussion on the notion of attribution can be found in BT and in Bishop (1995a, 1995b).

Here we show that the electrostatics analogy applies to highly accurate estimates of the PV such as those based on non-linear balance. In section 2 the general theoretical framework is established. Results are then summarized in section 3 in the simple case of geostrophic flow on an f-plane (section 2 can be omitted by readers content with the summary of section 3). In section 4 a comparison is made with the quasi-geostrophic formulation given in BT. The symmetries of the semi-geostrophic system permit elegant analytic solutions, presented in section 5, from which a precise description of non-linear bound charge has been made.

2. PV AND ELECTROSTATICS—BEYOND QUASI-GEOSTROPHY

The purpose of this section is to demonstrate the analogy between electrostatics and PV for forms of the PV which are considerably more accurate than the quasi-geostrophic approximation. As we shall show, an analogy to electrostatics can be made whenever the Ertel–Rossby PV can be accurately approximated with the form,

$$\rho \mathrm{PV} = \frac{\theta_0}{g} \nabla \cdot \left[-\frac{\partial^2 \psi}{\partial x \partial z} \frac{\partial \phi}{\partial z}, -\frac{\partial^2 \psi}{\partial y \partial z} \frac{\partial \phi}{\partial z}, (f + \nabla_{\mathrm{h}}^2 \psi) \frac{\partial \phi}{\partial z} \right] \tag{1}$$

where ϕ is the geopotential, ψ is the streamfunction for a non-divergent part of the flow, ρ is density, f is the Coriolis parameter, and θ_0 is a constant potential temperature. The hydrostatic approximation has been made in Eq. (1), viz $\partial \phi / \partial z = g\theta/\theta_0$; this relationship for θ results from making the quasi-Boussinesq approximation in which a reference density, depending on height alone, is used. As discussed by Xu (1994), the flow approximation is accurate whenever the rate of change of the vorticity associated with the divergent part of the flow together with its advection, tilting and stretching, can be neglected. This is a good approximation for synoptic scale systems. Xu argues that smaller scale meteorological phenomena such as fronts, curved fronts and some mesoscale vortices should also be well represented by this type of approximation. Raymond and Jiang (1990) and Raymond (1992) have shown, by direct simulation, that models which use PV in the form given by Eq. (1) can produce realistic descriptions of some types of mesoscale convective systems.

To define anomalies of PV it is necessary to subtract from the local PV an appropriate background distribution. We decompose the fields into a geostrophically balanced basic state flow, indicated by an overbar, dependent on latitude, y, and height, z, in an arbitrary way and a deviation from the basic state, indicated by a prime: $\phi = \bar{\phi}(y, z) + \phi'(x, y, z, t)$ and $\psi = \bar{\psi}(y, z) + \psi'(x, y, z, t)$. The basic state mass-weighted PV is then given by

$$\rho\overline{\mathrm{PV}} = \frac{\theta_0}{g}\nabla \cdot \left[0, -\frac{\partial^2\bar{\psi}}{\partial y \partial z}\frac{\partial\bar{\phi}}{\partial z}, \bar{\zeta}\frac{\partial\bar{\phi}}{\partial z}\right] \tag{2}$$

where $\bar{\zeta} = f + \partial^2\bar{\psi}/\partial y^2$.

To proceed it is convenient to define a reference mass-weighted potential vorticity, $\rho_0\mathrm{PV}_0 = f_0 N_0^2 \theta_0/g$, where subscript 0 indicates a constant value and N is the Brunt-Väisälä frequency. To introduce the electrostatics analogy we use these definitions to define an anomaly of potential vorticity charge density as:

$$e'_{\mathrm{f}} = f_0\rho(\mathrm{PV} - \overline{\mathrm{PV}})/(\rho_0\mathrm{PV}_0) \tag{3}$$

Note that it is the mass-weighted potential vorticity, ρPV, that appears in this definition. Haynes and McIntyre (1990) refer to this as the amount of PV substance per unit volume and hence it satisfies the conservation properties mentioned there. It is consistent with Haynes and McIntyre's definition of PV substance to refer to e'_{f} as the free PV charge density. The subscript f indicates that this part of the PV field is analogous to the free charge in electrostatics; i.e. it is this part of the mass-weighted PV field that can be moved by advective and other fluxes along isentropes; cf. Haynes and McIntyre (1990).

The definition of the 'free' potential vorticity charge density in Eq. (3) is in accord with the original definition of potential vorticity given in Rossby (1940). In this context Rossby's potential vorticity would be defined as $f_0(\mathrm{PV}/\mathrm{PV}_0 - 1)$; i.e. it is the relative isentropic vorticity the air would have if returned to a standard latitude and static stability. It has the advantage of having the dimensions of vorticity, as perhaps is appropriate for a 'potential' vorticity. Transfer of the name potential vorticity, coined by Rossby, to the vorticity invariant $(1/\rho)\boldsymbol{\zeta} \cdot \nabla\theta$, defined by Ertel (1942), appears to have occurred following the publication of Reed (1955).

For attribution it is important to describe how e'_{f} is inverted to find that component of the flow and temperature that can be associated with the PV anomaly. In order to invert e'_{f}, one must have a relationship between ψ' and ϕ'. Such a relationship is provided by the divergence equation. Here an approximate version of the divergence equation is utilized consistent with the assumption of balanced flow, (e.g. Xu 1994; Davis 1992; Raymond 1992; Gent and McWilliams 1983). Note, however, that if the rate of change of divergence is known, there seems to be no reason why the PV cannot be inverted using the unapproximated divergence equation (Bates *et al.* 1995). Assuming that the basic state is geostrophically balanced, the divergence equation can be written in the form:

$$\nabla_{\mathrm{h}}^2\phi' = \nabla_{\mathrm{h}}^2(f\psi') + \nabla \cdot (\mathbf{F}_1 + \mathbf{F}_2) \tag{4}$$

where $\mathbf{F}_1$ and $\mathbf{F}_2$ represent those terms which are linear and non-linear in ψ', respectively, and ∇_{h}^2 represents the horizontal Laplacian operator. Symbolically the inversion of (4) can be written as:

$$\phi' = f\psi' + \phi'_{\mathrm{a}} \quad \text{where} \quad \phi'_{\mathrm{a}} = \phi'_1 + \phi'_2 \tag{5a}$$

and

$$\phi'_1 = \nabla_{\mathrm{h}}^{-2}(\nabla \cdot \mathbf{F}_1) \quad \text{while} \quad \phi'_2 = \nabla_{\mathrm{h}}^{-2}(\nabla \cdot \mathbf{F}_2) \tag{5b}$$

In this equation ϕ'_1 and ϕ'_2 represent parts of the geopotential field which have a linear and non-linear dependence on ψ', respectively. The ∇_{h}^{-2} symbol represents the inverse horizontal Laplacian operator. Using (5) in (3) and grouping linear and non-linear terms, one finds that

$$e'_{\mathrm{f}} = \frac{1}{N_0^2}\nabla \cdot \left[-\frac{\partial^2\psi'}{\partial x \partial z}\frac{\partial\bar{\phi}}{\partial z}, -\frac{\partial^2\psi'}{\partial y \partial z}\frac{\partial\bar{\phi}}{\partial z} - \frac{\partial^2\bar{\psi}}{\partial y \partial z}\frac{\partial}{\partial z}(f\psi' + \phi'_1), \zeta'\frac{\partial\bar{\phi}}{\partial z} + \bar{\zeta}\frac{\partial}{\partial z}(f\psi' + \phi'_1) \right]$$
$$+ \text{ the divergence of non-linear terms in } \psi' \qquad (6)$$

where $\zeta' = \nabla_{\mathrm{h}}^2\psi'$.

The main aim in this paper is to establish a framework in which the field associated with each element of PV charge is an invariant quantity; that is, it is independent of the properties of the medium such as the static stability profile. Since electrostatics is formulated in this way, it is used here as a guide. The property of invariant attribution is attained in electrostatics by insisting that the operator defining total charge is an invariant quantity. The free space Green's functions* associated with this operator are themselves invariant and hence make the field attributed to each element of charge an invariant quantity. An important point to recognize is that the total charge is only equal to the free charge in the idealized situation of a charge in free-space; i.e. the part of the charge field that can be moved around in a conservative manner is generally not associated with an invariant operator. Thus, to make the notion of action-at-a-distance as clear in PV thinking as it is in electrostatics, we need to define a total charge of PV substance, e'_{t}. This is defined by an invariant operator which is equivalent to the free charge of PV substance, e'_{f}, in some idealized circumstance. An idealized circumstance which reduces the operator defining e'_{f} to an invariant occurs if:

(a) $\bar{\psi}$ and $\bar{\phi}$ are independent of x and y with $\partial^2\bar{\phi}/\partial z^2 = N_0^2$ a constant,
(b) the Coriolis parameter is constant; i.e. $f = f_0$,
(c) ϕ' and ψ' are in geostrophic balance; i.e. $\phi'_1 = \phi'_2 = 0$,
(d) the non-linear terms are negligible.

This idealized background state has, in fact, a PV equal to the reference (constant) PV_0 defined earlier so letting $\bar{\psi}_0$ and $\bar{\phi}_0$ define this state and substituting into (6) gives the definition of total charge density; viz,

$$e'_{\mathrm{t}} = \frac{1}{N_0^2}\nabla \cdot \left[-\frac{\partial^2\psi'}{\partial x \partial z}\frac{\partial\bar{\phi}_0}{\partial z}, -\frac{\partial^2\psi'}{\partial y \partial z}\frac{\partial\bar{\phi}_0}{\partial z}, \zeta'\frac{\partial\bar{\phi}_0}{\partial z} + f_0^2\frac{\partial\psi'}{\partial z} \right]$$
$$= \nabla_{\mathrm{h}}^2\psi' + \frac{f_0^2}{N_0^2}\frac{\partial^2\psi'}{\partial z^2} \qquad (7)$$

Remarkably, the total PV charge density for highly accurate forms of the PV is equivalent to the total charge PV density given in BT. † This shows that the same electrostatics analogy applies to both quasi-geostrophic and Ertel–Rossby potential vorticities. The Green's function for the operator defining (7) implies that the part of the streamfunction field, $\delta\psi$, attributable to each element, $e'_{\mathrm{t}}\delta V$, of the PV substance contained within an infinitesimal volume δV is

$$\delta\psi = -\frac{1}{4\pi}\left(\frac{e'_{\mathrm{t}}\delta V}{r}\right)$$

* By free space Green's function, we mean the Green's function subject to the boundary condition that all derivatives of the state variable diminish to zero at infinity. Also, recall that the operator defining total charge in electrostatics is the three-dimensional Laplacian and that the state variable which is operated on is the electric potential. For the type of PV discussed here the state variable is the streamfunction, ψ.

† In fact the quasi-geostrophic form has the complication of including the air density within the vertical derivatives. The Ertel–Rossby PV substance has no such complication.

where $r^2 = (x - x_0)^2 + (y - y_0)^2 + [(N_0/f_0)(z - z_0)]^2$ and (x_0, y_0, z_0) is the position of the element. Thus, $e'_t \delta V$ gives the total charge in an elemental volume and $\delta\psi$ gives the field induced by that element of charge. As in electrostatics the difference between the total charge and the free charge is the bound charge $e'_b = e'_t - e'_f$. Subtracting (6) from (7) gives

$$e'_b = e'_{bL} + e'_{bN}, \tag{8a}$$

where e'_{bL} and e'_{bN} are linear and non-linear functions of ψ', respectively. The algebraic expressions for these quantities are

$$e'_{bL} = \frac{1}{N_0^2}\nabla \cdot \Bigg[-\frac{\partial^2\psi'}{\partial x \partial z}\left(\frac{\partial\bar{\phi}_0}{\partial z} - \frac{\partial\bar{\phi}}{\partial z}\right), -\frac{\partial^2\psi'}{\partial y \partial z}\left(\frac{\partial\bar{\phi}_0}{\partial z} - \frac{\partial\bar{\phi}}{\partial z}\right) + \frac{\partial^2\bar{\psi}}{\partial y \partial z}\frac{\partial}{\partial z}(f\psi' + \phi'_1),\ \zeta'\left(\frac{\partial\bar{\phi}_0}{\partial z} - \frac{\partial\bar{\phi}}{\partial z}\right) + (f_0^2 - \bar{\zeta} f)\frac{\partial\psi'}{\partial z} - \bar{\zeta}\frac{\partial\phi'_1}{\partial z}\Bigg] \tag{8b}$$

and

$$\begin{aligned} e'_{bN} = & -\frac{1}{N_0^2}\nabla \cdot \left[-\frac{\partial^2\psi'}{\partial x \partial z} f\frac{\partial\psi'}{\partial z}, -\frac{\partial^2\psi'}{\partial y \partial z} f\frac{\partial\psi'}{\partial z}, \zeta' f\frac{\partial\psi'}{\partial z}\right] \\ & -\frac{1}{N_0^2}\nabla \cdot \left[0, -\frac{\partial^2\bar{\psi}}{\partial y \partial z}\frac{\partial\phi'_2}{\partial z}, \bar{\zeta}\frac{\partial\phi'_2}{\partial z}\right] \\ & -\frac{1}{N_0^2}\nabla \cdot \left[-\frac{\partial^2\psi'}{\partial x \partial z}\frac{\partial\phi'_a}{\partial z}, -\frac{\partial^2\psi'}{\partial y \partial z}\frac{\partial\phi'_a}{\partial z}, \zeta'\frac{\partial\phi'_a}{\partial z}\right] \end{aligned} \tag{8c}$$

At this point, note that the vectors defined by the square brackets in Eqs. (7), (8b) and (8c) are, respectively, the counterparts of the electric field and polarization vectors used in electrostatics. In electrostatics these vectors can always be defined in terms of some combination of the components of the gradient of electric potential. However, the above expressions are in terms of the components of the vorticity vector and the perturbation potential temperature and not in terms of the components of the gradient of the state variable ψ'. Given the importance of the vorticity vector in potential vorticity, this is not surprising. However, the divergence of the vorticity vector is zero and it turns out that (7) and (8) can be written in terms of the divergence of a vector whose components are made up of the components of the gradient of the streamfunction and the vertical derivative of ϕ'_a. The alternative expression for the total charge is

$$e'_t = \nabla \cdot \left(\frac{\partial\psi'}{\partial x}, \frac{\partial\psi'}{\partial y}, \frac{f_0^2}{N_0^2}\frac{\partial\psi'}{\partial z}\right) \tag{9}$$

Here, the term in brackets gives the counterpart of the electric field vector.

For the bound charge, Eq. (8c) already gives the non-linear part of the bound charge in terms of products involving first order derivatives of ψ', ϕ'_2 and ϕ'_a. The linear part, e'_{bL}, can also be expressed in such a form. This involves rewriting Eq. (8b) in the following way (which is not obvious at first sight):

$$e'_{\mathrm{bL}} = \frac{1}{N_0^2}\nabla \cdot \left[(N_0^2 - \bar{N}^2)\frac{\partial \psi'}{\partial x}, (N_0^2 - \bar{N}^2)\frac{\partial \psi'}{\partial y} + \frac{\partial^2 \bar{\psi}}{\partial y \partial z}\frac{\partial}{\partial z}(f\psi' + \phi'_1), \right.$$
$$\left. (f_0^2 - \bar{\zeta} f)\frac{\partial \psi'}{\partial z} + \frac{\partial^2 \bar{\phi}}{\partial y \partial z}\frac{\partial \psi'}{\partial y} - \bar{\zeta}\frac{\partial \phi'_1}{\partial z}\right] \tag{10}$$

where $\bar{N}^2 = \partial^2\bar{\phi}/\partial z^2$ is the static stability of the background state in which the PV anomaly is situated. Here, the term in the square brackets gives the negative of the counterpart to the polarization vector. Note that provided N_0 is less than or equal to $\bar{N}$ the horizontal part of the polarization vector will tend (provided $\partial^2\bar{\psi}/\partial y\partial z$ is small) to point in the same direction as the electric field vector. Thus, provided N_0 is chosen in this way, the bound charge at a point will tend to be of the opposite sign to both the total and free charge. Since the free charge is just the difference between the total and bound charge, it is also expressible in terms of first-order derivatives of ψ', ϕ'_2 and ϕ'_{a}. Its full form is rather cumbersome to write down but in the special case of geostrophic balance on an f-plane, it takes on a simpler form which will be explored in the next section.

3. Geostrophic balance on an f-plane

Here a summary of the formulation is given making the assumption, for simplicity, that the flow associated with the PV anomaly is in geostrophic balance on an f-plane. In this case the free PV charge, as defined in Eq. (3), becomes:

$$e'_{\mathrm{f}} = e'_{\mathrm{fL}} + e'_{\mathrm{fN}} \tag{11a}$$

where

$$e'_{\mathrm{fL}} = \frac{1}{N_0^2}\nabla \cdot \left[\bar{N}^2\frac{\partial \psi'}{\partial x}, \bar{N}^2\frac{\partial \psi'}{\partial y} - f_0\frac{\partial^2 \bar{\psi}}{\partial y \partial z}\frac{\partial \psi'}{\partial z}, \bar{\zeta} f_0\frac{\partial \psi'}{\partial z} - f_0\frac{\partial^2 \bar{\psi}}{\partial y \partial z}\frac{\partial \psi'}{\partial y}\right] \tag{11b}$$

and

$$e'_{\mathrm{fN}} = \frac{f_0}{N_0^2}\nabla \cdot \left[-\frac{\partial^2 \psi'}{\partial x \partial z}\frac{\partial \psi'}{\partial z}, -\frac{\partial^2 \psi'}{\partial y \partial z}\frac{\partial \psi'}{\partial z}, \zeta'\frac{\partial \psi'}{\partial z}\right]. \tag{11c}$$

The terms in square brackets define the displacement vector for geostrophically balanced PV fields. From these vectors, we may write down the dielectric tensor for PV fields, i.e. $e'_{\mathrm{f}} = \nabla \cdot \mathbf{D}'$, where the vector $\mathbf{D}'$ is defined as:

$$\mathbf{D}' = (\boldsymbol{\varepsilon}_{\mathrm{L}} + \boldsymbol{\varepsilon}_{\mathrm{N}})\nabla\psi' \tag{12}$$

where $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{\mathrm{L}} + \boldsymbol{\varepsilon}_{\mathrm{N}}$ is the dielectric tensor with components ε_{ij} with i and j ranging from 1 to 3. The linear and non-linear components have been separated viz: $e'_{\mathrm{fL}} = \nabla \cdot (\boldsymbol{\varepsilon}_{\mathrm{L}}\nabla\psi')$ and $e'_{\mathrm{fN}} = \nabla \cdot (\boldsymbol{\varepsilon}_{\mathrm{N}}\nabla\psi')$. From inspection of Eq. (11b) the components of the linear part of the dielectric tensor, $\boldsymbol{\varepsilon}_{\mathrm{L}}$, are:

$$\boldsymbol{\varepsilon}_{\mathrm{L}}\text{: } \varepsilon_{11} = \varepsilon_{22} = \frac{\bar{N}^2}{N_0^2},\ \varepsilon_{33} = \frac{f_0\bar{\zeta}}{N_0^2},$$
$$\varepsilon_{23} = \varepsilon_{32} = \frac{f_0\bar{u}_z}{N_0^2} \quad \text{and other components zero} \tag{13a}$$

The only non-zero part of the non-linear component, $\boldsymbol{\varepsilon}_{\mathrm{N}}$, is proportional to the relative vorticity vector of the anomaly i.e.

$$\boldsymbol{\varepsilon}_{\mathrm{N}}: \quad \varepsilon_{i3} = \frac{f_0}{N_0^2}\left(-\frac{\partial^2\psi'}{\partial x \partial z}, -\frac{\partial^2\psi'}{\partial y \partial z}, \nabla_{\mathrm{h}}^2\psi'\right) \quad \text{and other components zero} \tag{13b}$$

Thus the linear part only depends on the background static stability and flow. However, the non-linear part depends on the geopotential itself via the relative vorticity; this is not known until the inversion solution has been obtained.

Following the electrostatics analogy we have defined a total PV charge, e_{t}', to be independent of the properties of the basic-state atmosphere: $e_{\mathrm{t}}' = \nabla \cdot (\boldsymbol{\varepsilon}_0 \nabla\psi')$. Here $\boldsymbol{\varepsilon}_0$ is the equivalent of the permittivity of 'free-space' and is defined from Eq. (9) by the following components:

$$\boldsymbol{\varepsilon}_0: \quad \varepsilon_{11} = \varepsilon_{22} = 1, \, \varepsilon_{33} = \frac{f_0^2}{N_0^2}, \quad \text{and other components zero} \tag{14}$$

The definition of the bound PV, $e_{\mathrm{b}}' = e_{\mathrm{t}}' - e_{\mathrm{f}}'$, equivalent to the polarized charge in electrostatics is therefore:

$$e_{\mathrm{b}}' = -\nabla \cdot [(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_0)\nabla\psi'] \tag{15}$$

In the quasi-geostrophic case, as described by BT, the bound PV was due to the equivalent of a vertical polarization (as $\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_0$ is only non-zero in the vertical component; see section 4(a) below). It can be seen that in general the Ertel–Rossby case bound PV is due to a linear contribution with both horizontal *and* vertical polarization and a new non-linear contribution associated with e_{fN}'. Therefore, the non-linear parts of the PV are, in this electrostatics analogy, components of the bound charge. This provides an attractive physical picture of the differences between the fields attributable to Ertel–Rossby and quasi-geostrophic forms of the PV.

The field induced by the non-linear part of the bound charge will now be shown to rapidly diminish with distance away from an isolated free PV charge. The net non-linear bound charge in a volume is equal to the volume integral of e_{bN}'. By Gauss's theorem, this integral is equal to the flux of the polarization field through the surface containing the volume. Suppose that the amplitude of ψ' was inversely proportional to the distance r away from a localized free PV anomaly. Then, using the fact that $e_{\mathrm{bN}}' = -e_{\mathrm{fN}}'$ (as described in section 2) and using Eq. (11c), the non-linear part of the polarization field would be proportional to $1/r^5$. Consequently, the bound charge contained in a sphere of radius r enclosing the anomaly would be proportional to $1/r^3$. Thus, the total bound charge associated with this isolated free PV anomaly is zero and hence its effect on the far-field is zero. Note also that, in this case, the amplitude of the bound charge would be proportional to $1/r^6$. This qualitative discussion suggests that the effect of the non-linear part of the bound charge on the streamfunction field is localized near isolated anomalies and may not affect the far-field features of the streamfunction field.

Note that the above argument applies equally well to the less approximated form of the non-linear part of the bound charge given in Eq. (8b). In section 5, we illustrate this localization of non-linear bound charge with an example from semi-geostrophic theory.

4. Comparison with the quasi-geostrophic PV

(a) *Dielectric tensor*

A simplification to the general theory described in the previous section is provided by assuming a background PV which is only a function of height. Then there is a background potential vorticity $\overline{\mathrm{PV}} = f\bar{N}^2\theta_0/g\rho$, where $\bar{N}$ is the background Brunt–Väisälä frequency which is taken as a function of height only. The only changes to the equations described in the previous section are that: $\bar{N}^2 = \bar{N}^2(z)$, $\bar{\zeta} = f$, and $\bar{\psi}_{yz} = 0$. Therefore the dielectric tensor linear part becomes:

$$\boldsymbol{\varepsilon}_{\mathrm{L}}: \quad \varepsilon_{11} = \varepsilon_{22} = \frac{\bar{N}_2}{N_0^2},\ \varepsilon_{33} = \frac{f_0^2}{N_0^2} \quad \text{and other components zero} \tag{16}$$

The non-linear dielectric tensor, $\boldsymbol{\varepsilon}_{\mathrm{N}}$, is given by Eq. (13b).

Compare this dielectric tensor for the Ertel–Rossby PV with that obtained in BT for the quasi-geostrophic PV. It is convenient here to take the QG system assuming constant density. Using the same coordinates as above the relevant quasi-geostrophic forms are: $q_{\mathrm{f}}' = \nabla \cdot \mathbf{D}_{\mathrm{Q}}'$ where $\mathbf{D}_{\mathrm{Q}}' = \boldsymbol{\varepsilon}\nabla\psi'$ and the quasi-geostrophic dielectric tensor only has a linear part with components given by:

$$\mathrm{QG}: \quad \varepsilon_{11} = \varepsilon_{22} = 1,\ \varepsilon_{33} = \frac{f_0^2}{\bar{N}^2} \quad \text{and other components zero} \tag{17}$$

It is clear by comparing the expressions in Eqs. (16) and (17) for the dielectric constant that the quasi-geostrophic PV is *not*, in general, equal to the linear part of the Ertel–Rossby PV. They only become equal in the case when there is no vertical variation in the background static stability so that we may define $\bar{N}^2 = N_0^2$ and Eqs. (16) and (17) are then identical. This difference is highlighted by consideration of the bound PV in the two cases. For the QG system polarization only occurs in the vertical direction whereas in the Ertel–Rossby case linear polarization only occurs in the horizontal directions. This shows the power of the electrostatics analogy in giving a way of visualizing the linear part of the difference between Ertel–Rossby and QG PV. The implications of this difference are considered in the next section for the problem of a point PV anomaly residing beneath the tropopause; the solution in the quasi-geostrophic case having been given in BT.

(b) *The role of the tropopause*

An interesting case arises when there is a point anomaly, Q, of perturbation free-charge density beneath the tropopause with a jump in the static stability. To simplify the analysis, assume that Q is of small enough amplitude and is sufficiently distant from the tropopause that in the vicinity of the tropopause the effect of the non-linear bound charge in the troposphere is zero.

In order that there be no free charge at the tropopause the vertical component of the displacement vector defined in Eq. (11b) must be continuous across the tropopause. With ψ' continuous across the tropopause, this condition is met when the potential temperature, and hence ψ_z', is continuous across the boundary. (Note that the boundary condition imposed at the tropopause in the QG case, to give zero e_{fL}' at the interface, is different: continuity of ψ_z'/N^2. The QG boundary condition implies an unphysical jump in potential temperature at the tropopause. This jump is removed in the Ertel–Rossby case). For a point charge (at

$z = 0$) a distance a below the tropopause (at $z = a$), these conditions are satisfied if

$$\begin{aligned}\psi_t' &= -\frac{Q}{4\pi}\left(\frac{1}{(x^2+y^2+z^2N_t^2/f^2)^{1/2}} + \frac{n}{(x^2+y^2+(z-2a)^2N_t^2/f^2)^{1/2}}\right)\\ \psi_s' &= -\frac{Q(1+n)}{4\pi(x^2+y^2+(z-la)^2N_s^2/f^2)^{1/2}}\end{aligned} \tag{18}$$

where $n = -l/m$ where $l = (1 - N_t/N_s)$ and $m = (1 + N_t/N_s)$. The coordinates (x, y, z) indicate the distance from the point charge. The subscripts are t for troposphere and s for stratosphere.

The solution in Eq. (18) is identical to that for a free charge of QG PV beneath the tropopause, given in Eq. (12) of BT, *except* that here n has the opposite sign. One of the important features of Eq. (18) is that it is the solution for an element of free charge at large distances from the anomaly due to the rapid diminution of the effects of the non-linear terms at large distance. For example for $z \to \infty$ the limit is $\psi_s' \to -Q(1 - l/m)/4\pi r_s$ whereas in the quasi-geostrophic case discussed in BT the equivalent limit is given by $\psi_s' \to -Q(1 + l/m)/4\pi r_s$; where $r_s = (x^2 + y^2 + z^2N_s^2/f^2)^{1/2}$.

The electrostatics analogy can be used to provide a physical picture of why there is a far-field difference between the QG and the Ertel–Rossby solutions for the tropopause problem posed in this section. As discussed in BT the bound, or polarized, PV charge plays a crucial role in interpreting the solution. Polarization is governed by the susceptibility tensor, $\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_0$. In the QG system the susceptibility is proportional to $(N_0^2 - \bar{N}^2)/\bar{N}^2$ whereas for the Ertel–Rossby PV the linear part of the susceptibility is proportional to $(\bar{N}^2 - N_0^2)/N_0^2$.* Therefore in the QG system, the troposphere is more susceptible to polarization than the stratosphere. Thus if a positive free PV anomaly is introduced into the troposphere, positive bound charge concentrates at the tropopause. Conversely in the Ertel–Rossby system the troposphere is *less* susceptible to polarization than the stratosphere. Hence a positive free charge in the troposphere gives *negative* bound charge at the tropopause. Therefore, as is evident from this comparison between the QG and Ertel–Rossby solutions, these differences in the bound charge significantly affect the far-field solution.

Another important difference between QG and Ertel–Rossby PV is apparent in the limit of infinite stability in one of the media. The solution given in Eq. (18) also applies, for example, to an anomaly (at $z = 0$) above the Earth's surface (at $z = -a$) if $a \to -a$ in Eq. (18) and we take the limit $N_s^2 \to \infty$ to mimic the rigid aspect of the surface (in this case s stands for subterranean). In this limit $\psi' = 0$ but $\psi_z' \propto \theta' \neq 0$ at the surface for the Ertel–Rossby solution. However for the QG solution (Eq. (12) of BT) $\psi' \neq 0$ but $\psi_z' \propto \theta' = 0$. Attribution, using piecewise potential vorticity inversions, is often done by assuming $\theta' = 0$ at the surface. This analysis suggests that $\psi' = 0$ may be an appropriate condition at the Earth's surface for Ertel–Rossby PV anomalies. With this condition, surface pressure falls are seen as a consequence of vortex stretching resulting from ascent ahead of an advecting PV anomaly rather than being linked *per se* by PV inversion to that anomaly.

5. Inversion solutions for an Ertel–Rossby PV anomaly

In this section the case of a single PV anomaly embedded in a uniform background PV with no flow is considered as it provides analytical solutions which make the previous

* The components of the susceptibility tensor must be positive for like charges to repel and unlike charges to attract each other. As the constant N_0^2 is arbitrary it can be chosen differently in the QG and Ertel–Rossby cases to ensure a positive susceptibility.

theoretical formulation explicit. For this situation we can take $\bar{N}^2 = N_0^2$, $\overline{\mathrm{PV}} = \overline{\mathrm{PV}}_0$, and $\rho = \rho_0$ as constants. In this case the linear part of the bound charge density $e'_{\mathrm{bL}} = 0$ (see Eq. (10)), $e'_{\mathrm{bN}} = -e'_{\mathrm{fN}}$, and $e'_{\mathrm{fL}} = e'_{\mathrm{t}}$. Here we exploit an important mathematical simplification that arises if the PV anomaly is spherically symmetric in the radial coordinate, r, defined following Eq. (7). The geostrophic flow assumption will be made by setting $\psi' = \phi'/f_0$.

(*a*) *Quasi-geostrophic case*

The quasi-geostrophic potential vorticity, q', can be expressed, using the radial coordinate transformation, as:

$$q' = \frac{1}{f_0 r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi'}{\partial r}\right) \tag{19}$$

This shows that the quasi-geostrophic PV is, in this case, equal to the linear part of the Ertel–Rossby PV, e'_{fL}. The magnitude of the anomalies are equal if $q' = e'_{\mathrm{fL}} = (\mathrm{PV} - \mathrm{PV}_0) f_0/\mathrm{PV}_0$.

A simple example of such a PV anomaly is a 'ball' which has a constant but different PV compared to the rest of the atmosphere:

$$\begin{aligned} \text{Ball solution:} \quad & r > b \quad q' = 0 \quad \Rightarrow \phi' = -\frac{f_0}{3}\tilde{q}\frac{b^3}{r} \\ & r < b \quad q' = \tilde{q} \quad \Rightarrow \phi' = -f_0 \tilde{q}\left(\frac{b^2}{2} - \frac{r^2}{6}\right) \end{aligned} \tag{20}$$

Such a ball anomaly of PV represents, in some senses, an ideal shaped anomaly in that it has equal dimensions in all three directions as long as the normal scaling of the vertical coordinate (N/f) is made as befits the usual shallow atmosphere approximation. As mentioned by Hoskins *et al.* (1985) the vorticity anomaly is related to the static stability anomaly in the following way:

$$\begin{aligned} r < b \quad & \frac{\zeta'}{f_0} = 2\frac{\partial\theta'/\partial z}{\partial\bar{\theta}/\partial z} = \frac{2\tilde{q}}{3 f_0} \\ r > b \quad & \frac{\zeta'}{f_0} = -\frac{\partial\theta'/\partial z}{\partial\bar{\theta}/\partial z} \end{aligned} \tag{21}$$

Hence the vorticity anomaly inside the PV anomaly is of *twice* the magnitude of the static stability anomaly *irrespective* of the PV magnitude or of the anomaly dimension b. This partition will change, however, as the anomaly changes shape. As Hoskins *et al.* (1985) note, as the PV anomaly becomes tall and narrow a greater proportion is partitioned into vorticity whereas for a wide and shallow anomaly a greater proportion is in the static stability.

(*b*) *Semi-geostrophic PV*

An alternative estimate of the Ertel–Rossby PV is given by the so-called semi-geostrophic potential vorticity, SPV, defined as:

$$\mathrm{SPV} = \frac{1}{\rho_0}\boldsymbol{\zeta}_{\mathrm{SG}} \cdot \nabla\theta = \frac{1}{\rho_0}\nabla \cdot (\theta \boldsymbol{\zeta}_{\mathrm{SG}}) \tag{22}$$

where

$$\zeta_{SG} = \zeta + \frac{1}{f_0}\mathbf{J}_{xyz}(u, v)$$

Note that here suffices x, y, and z refer to spatial derivatives, winds (u, v) are geostrophic, and $\mathbf{J}$ is the three-dimensional vector Jacobian.

The semi-geostrophic vorticity has additional Jacobian terms which do not appear in a simple geostrophic estimation of vorticity, Hoskins (1982). These are formally small in situations where the semi-geostrophic approximation is valid. However, following Shutts and Cullen (1987), it is possible to make progress whilst retaining the small quadratic terms and remaining in Cartesian coordinates. It is possible, as described in the Appendix, to obtain an expression for the semi-geostrophic free PV charge, $s'_f = (\text{SPV} - \overline{\text{SPV}})f_0/\overline{\text{SPV}}$, using the radial coordinate r:

$$s'_f = \frac{1}{f_0 r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi'}{\partial r} + \frac{1}{f_0^2}r\left(\frac{\partial\phi'}{\partial r}\right)^2 + \frac{1}{3f_0^4}\left(\frac{\partial\phi'}{\partial r}\right)^3\right) \tag{23}$$

Notice that the first term is linear and assumes the mathematical form of the quasi-geostrophic PV given in Eq. (19), the second term is quadratic, and the third term is cubic in form.

It can be shown that the solution of Eq. (23) for a localized ball of SPV with $s'_f = \tilde{s}$ for $r < b$ and $s' = 0$ for $r > b$ is given by the following:

$$\begin{aligned} \frac{\partial\phi'}{\partial r} &= f_0^2 r\left(\left(1 + \frac{\tilde{s}}{f_0}\frac{b^3}{r^3}\right)^{1/3} - 1\right) \quad \text{for } r > b \\ \frac{\partial\phi'}{\partial r} &= f_0^2 r\left(\left(1 + \frac{\tilde{s}}{f_0}\right)^{1/3} - 1\right) \quad \text{for } r < b \end{aligned} \tag{24}$$

This solution is related to that given by Shutts (1991) for the total flow due to a point anomaly. Here though the dependence of the flow on the strength of the PV anomaly is given explicitly. The temperature and flow can be found directly from $\partial\phi'/\partial r$.

Inside the ball anomaly it is possible to integrate Eq. (24) to obtain the geopotential itself:

$$\phi' = f_0^2\frac{r^2}{2}\left(\left(1 + \frac{\tilde{s}}{f_0}\right)^{1/3} - 1\right) + \phi'_{00} \quad \text{for } r < b$$

where

$$\begin{aligned} \phi'_{00} = -f_0^2\bigg(\int_b^\infty r\left(\left(1 + \frac{\tilde{s}}{f_0}\frac{b^3}{r^3}\right)^{1/3} - 1\right)\mathrm{d}r \\ + \frac{b^2}{2}\left(\left(1 + \frac{\tilde{s}}{f_0}\right)^{1/3} - 1\right)\bigg) \end{aligned} \tag{25}$$

Outside the anomaly, $r > b$, ϕ' can only be expressed as an integral but for large radii we can show that:

$$\begin{aligned} \text{For } r \to \infty \quad \frac{\partial\phi'}{\partial r} &\approx f_0\frac{\tilde{s}}{3}\frac{b^3}{r^2}\left(1 - \frac{1}{3}\frac{\tilde{s}}{f_0}\frac{b^3}{r^3} + \ldots\right) \\ \Rightarrow \phi' &\approx -f_0\frac{\tilde{s}}{3}\frac{b^3}{r}\left(1 - \frac{1}{12}\frac{\tilde{s}}{f_0}\frac{b^3}{r^3} + \ldots\right) \end{aligned} \tag{26}$$

So, by comparing Eqs. (26) and (20), the semi-geostrophic solution for large radius has the same radial dependence as the quasi-geostrophic one.

In Fig. 1 the semi-geostrophic and quasi-geostrophic solutions are compared when $s_f' = q' = 4f_0$. This strong positive anomaly exhibits the important feature that the quasi-geostrophic solution has local regions of static instability and so is outside its domain of validity. In contrast the semi-geostrophic solution does not exhibit static instability for any magnitude anomaly. It can be seen that inside the anomaly the QG solution has much larger static stability and vorticity than its semi-geostrophic counterpart. This is associated with the fact that the QGPV is additive in terms of vorticity and stability whereas the SPV is multiplicative. From Eq. (24) it is clear that the semi-geostrophic solution inside the anomaly exhibits exactly the same relationship between the (geostrophic) vorticity and static stability anomalies as the quasi-geostrophic case of Eq. (21). The connection of these anomalies to the magnitude of the PV itself can also be obtained from Eq. (24):

$$\frac{\zeta'}{f_0} = 2\frac{\partial\theta'/\partial z}{\partial\bar{\theta}/\partial z} = 2\left(\left(1+\frac{\tilde{s}}{f_0}\right)^{1/3} - 1\right) \quad \text{for } r < b$$

An approximation to this relationship in the region outside the PV anomaly can be obtained by ignoring the horizontal vorticity components (and those associated with the Jacobian term):

$$\frac{\zeta'}{f_0} \approx -\frac{\dfrac{\partial\theta'}{\partial z}\Big/\dfrac{\partial\bar{\theta}}{\partial z}}{1+\dfrac{\partial\theta'}{\partial z}\Big/\dfrac{\partial\bar{\theta}}{\partial z}} \quad \text{for } r > b$$

In contrast to the QG solution this latter relationship implies that anticyclonic vorticity is limited such that the absolute vorticity is positive.

(c) *Semi-geostrophic dielectric constant*

Using the terminology of section 2 Eq. (23) can be written in the following form:

$$s_f' = \nabla \cdot \mathbf{D}_s' \tag{27}$$

where the vector field associated with a semi-geostrophic PV anomaly written in the spherical coordinates is $\mathbf{D}_s' = \varepsilon_s(1/f_0)(\partial\phi'/\partial r)\hat{\mathbf{r}}$ where $\hat{\mathbf{r}}$ is a unit vector in the radial direction. The quantity ε_s, the semi-geostrophic dielectric constant, is a scalar in these coordinates and has the following form obtained by comparing Eqs. (23) and (27):

$$\varepsilon_s = 1 + \frac{1}{f_0^2 r}\frac{\partial\phi'}{\partial r} + \frac{1}{3}\left(\frac{1}{f_0^2 r}\frac{\partial\phi'}{\partial r}\right)^2 \tag{28}$$

It is clear that the SG dielectric constant is not unity even in the case being considered here in which the background static stability is uniform. Therefore as shown in section 3 there is a non-linear contribution to the bound charge. To complete the electrostatics analogy for semi-geostrophic theory the equivalent of the electric and polarization fields can be defined as follows:

$$s_f' + s_b' = \nabla \cdot \mathbf{E}_s' \quad \text{and} \quad \mathbf{E}_s' = \frac{1}{f_0}\frac{\partial\phi'}{\partial r}\hat{\mathbf{r}}$$

$$s_b' = -\nabla \cdot \mathbf{P}_s' \quad \text{and} \quad \mathbf{P}_s' = \chi_s\frac{1}{f_0}\frac{\partial\phi'}{\partial r}\hat{\mathbf{r}}$$

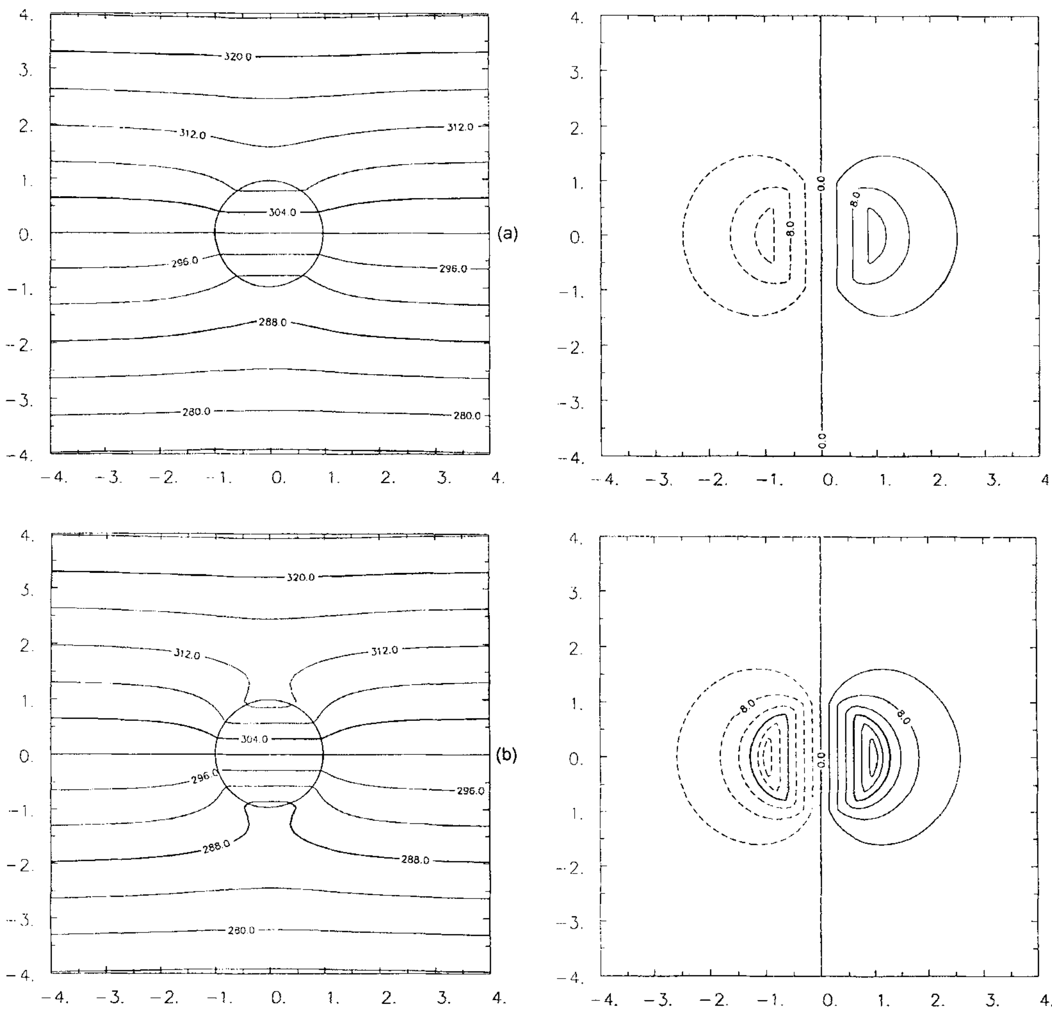

Figure 1. (a) Vertical section showing potential temperature (contour interval 4 K) and normal velocity (contour interval 4 m s^{-1}) for a semi-geostrophic PV anomaly of magnitude $\tilde{s}/f_0 = 4$ with a spherical shape in the stretched coordinate. Axes are marked with distances divided by the arbitrary anomaly radius. (b) As in (a) for a quasi-geostrophic anomaly of the same magnitude.

where $\chi_s = \varepsilon_s - 1$ and s_b' is the semi-geostrophic bound charge.

An explicit expression for the SG dielectric constant for the ball charge can be found using Eq. (24):

$$\begin{aligned}\varepsilon_s &= 1 + \tfrac{1}{3}\left(\left(1+\frac{\tilde{s}}{f_0}\right)^{1/3} - 1\right)\left(\left(1+\frac{\tilde{s}}{f_0}\right)^{1/3} + 2\right) \quad \text{for } r < b \\ \varepsilon_s &= 1 + \tfrac{1}{3}\left(\left(1+\frac{\tilde{s}}{f_0}\frac{b^3}{r^3}\right)^{1/3} - 1\right)\left(\left(1+\frac{\tilde{s}}{f_0}\frac{b^3}{r^3}\right)^{1/3} + 2\right) \quad \text{for } r > b\end{aligned} \tag{29}$$

In Fig. 2(a) the dependence of ε_s on $\tilde{s}$ for $r < b$ is given. It is interesting to note that the susceptibility, χ_s, has the same sign as the PV anomaly, $\tilde{s}$. In Fig. 2(b) the radial dependence

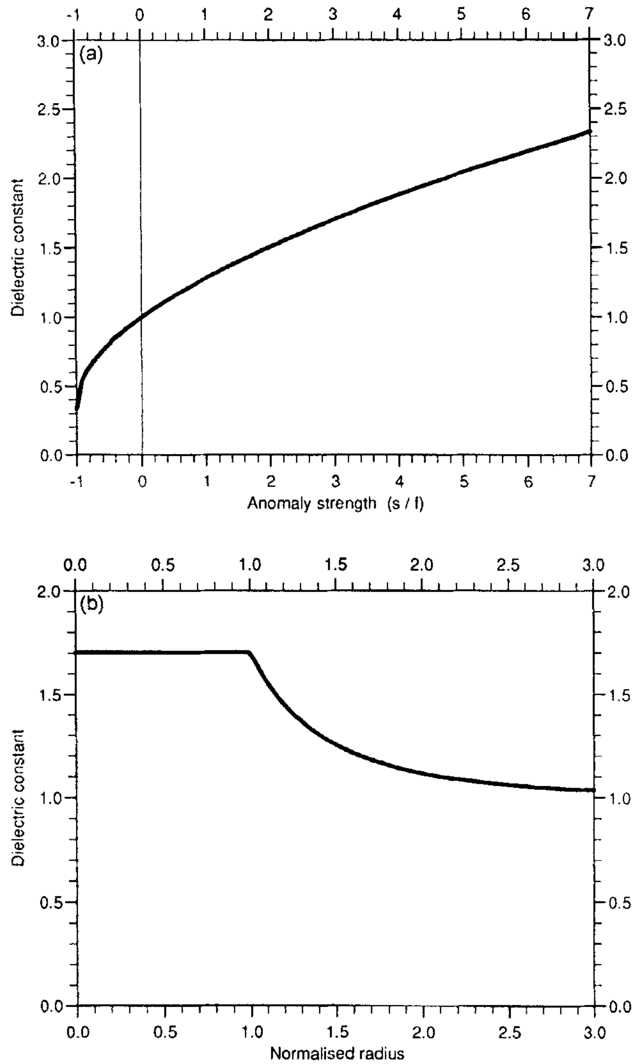

Figure 2. (a) The semi-geostrophic dielectric constant, ε_s, within the potential vorticity anomaly as a function of the anomaly strength $\tilde{s}/f_0$. (b) Radial variation of the semi-geostrophic dielectric constant for an anomaly of magnitude $\tilde{s}/f_0 = 3$. Radius is normalized by the anomaly radius.

of ε_s is shown for an anomaly with $\tilde{s} = 3f_0$. Note that ε_s rapidly approaches unity away from the PV anomaly.

The bound charge density, s_b', can be evaluated for the ball solution:

$$\left.\begin{aligned} s_b' &= -f_0\left(\left(1+\frac{\tilde{s}}{f_0}\right)^{1/3}-1\right)^2\left(\left(1+\frac{\tilde{s}}{f_0}\right)^{1/3}+2\right) \quad \text{for } r<b \\ s_b' &= 3f_0\left\{\frac{\left(1+\dfrac{2}{3}\dfrac{\tilde{s}}{f_0}\dfrac{b^3}{r^3}\right)}{\left(1+\dfrac{\tilde{s}}{f_0}\dfrac{b^3}{r^3}\right)^{2/3}}-1\right\} \quad \text{for } r>b \end{aligned}\right\} \tag{30}$$

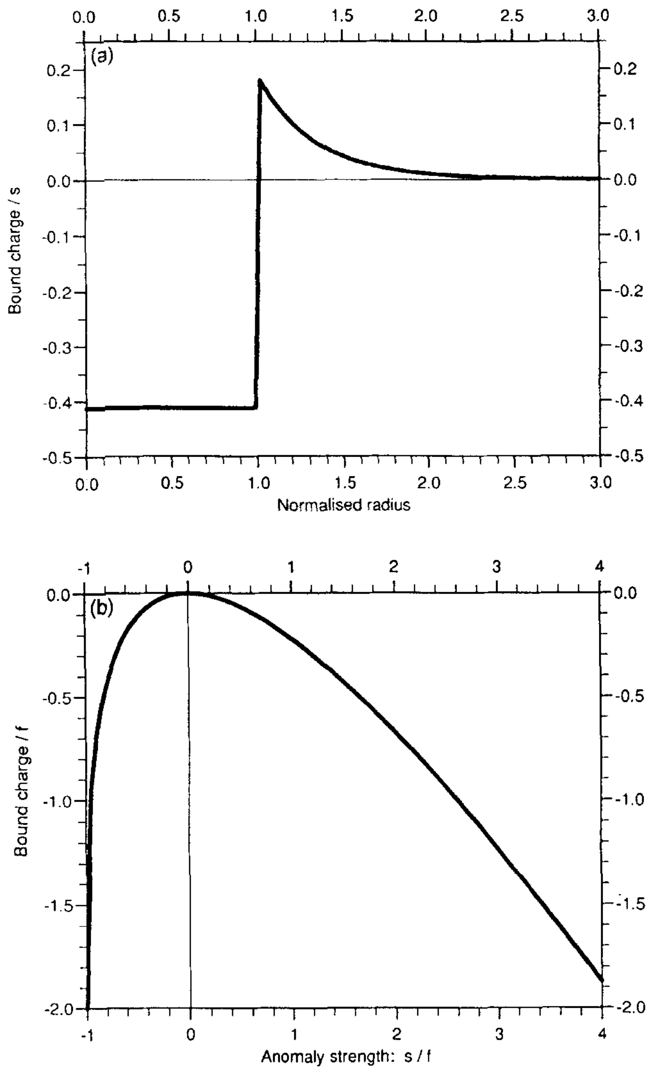

Figure 3. (a) Radial variation of the bound charge for an anomaly of strength $\tilde{s}/f_0 = 3$. The bound charge has been normalized by the anomaly magnitude. (b) The dependence of bound charge, within the PV anomaly, on the anomaly strength. The bound charge has been normalized by f_0.

Hence the sign of s_b' is independent of the sign of the PV anomaly. Within the ball s_b' is negative. Outside the ball it is positive and at large radii it has a $1/r^6$ variation. Note that the dependence of s_b' on the radius is consistent with the heuristic argument given at the end of section 3. Using Gauss's integral theorem, it is straightforward to prove that the volume integral of the bound charge density, s_b', is zero.

Figure 3 shows the form of the bound charge s_b' as a function of radius for a range of values of $\tilde{s}$. In Fig. 3(a) the radial dependence of $s_b'/\tilde{s}$ is given for an anomaly with $\tilde{s}/f_0 = 3$. The bound charge is therefore composed of a central negative PV anomaly surrounded by an equal amount of positive PV. Figure 3(b) gives the dependence of s_b' on the anomaly strength, $\tilde{s}$, inside the anomaly where s_b' is independent of radius. For the most extreme anticyclonic anomaly, $\tilde{s}/f_0 = -1$, with zero total PV inside the anomaly the bound charge $s_b'/f_0 = -2$. In this case the total charge is $(s_f' + s_b')/f_0 = -3$.

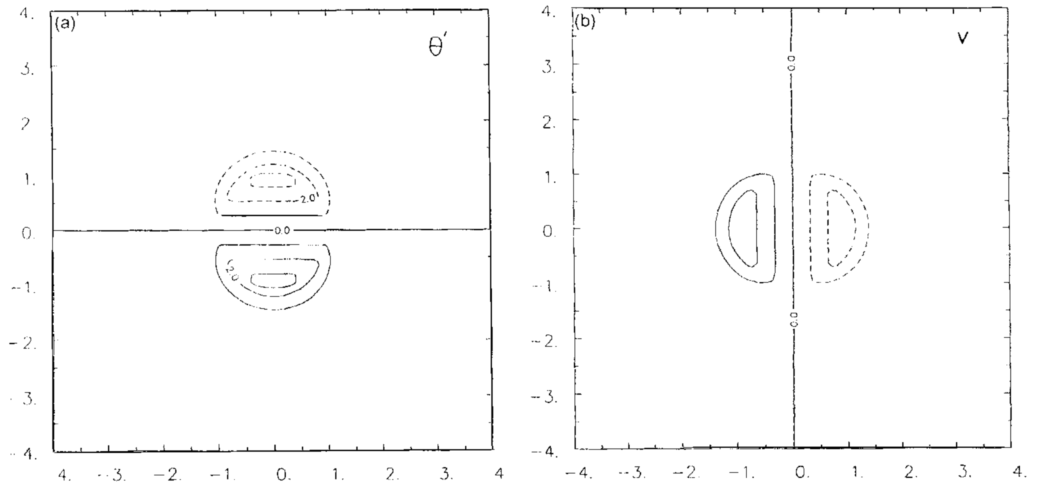

Figure 4. The difference in (a) potential temperature (contour interval 2 K) and (b) normal velocity (contour interval 4 m s^{-1}) between the semi-geostrophic and quasi-geostrophic solutions given in Fig. 1.

Differences between the SG and QG flow and temperature are given in Fig. 4 for the case described by Fig. 1. The difference is similar to that due to a negative anomaly located within the semi-geostrophic ball as can be seen by the anticyclonic flow and temperature dipole. This difference reflects the high concentration of negative bound charge within the SG ball.

As the semi-geostrophic geopotential tends to the quasi-geostrophic form at large radii the equivalent QG anomaly is $q' = s'_{\rm f}$. It is then clear that the reciprocal of the semi-geostrophic dielectric constant gives a measure of the modification to the QG field made in using the semi-geostrophic equations. From Fig. 2(b) it is clear that this modification is large only close to the anomaly.

6. Discussion

In this paper we have shown that the same electrostatics analogy applies to both the Ertel–Rossby potential vorticity and the quasi-geostrophic approximation to potential vorticity. For a PV anomaly embedded in an arbitrary zonally-averaged flow, with background PV varying with latitude and height, the analogy consists of describing the atmosphere in terms of a non-linear dielectric medium. The associated dielectric 'constant' is composed of a linear and a non-linear part. The linear part is similar but not exactly the same as in the quasi-geostrophic case. The difference exists if there is a vertical variation in the background PV such as occurs at the tropopause or if the zonal flow is sheared in the horizontal or vertical. In contrast close to an Ertel–Rossby PV anomaly the dielectric 'constant' has a significant non-linear contribution which produces streamfunction features markedly different to that associated with QG anomalies.

It is apparent that it is the amount of PV-substance per unit volume, ρPV, which is associated with the size of the induced flow and temperature. Hence a PV anomaly in the stratosphere has a much smaller effect than the same magnitude anomaly located in the troposphere because of the decrease of density with height. Lower tropospheric PV anomalies therefore take on an increased significance as a consequence of these ideas. The realization that it is the PV-substance per unit volume that is important in producing field also sheds light on the PV scaling problems, discussed by Lait (1994), associated with the observed exponential increase of PV with height.

The non-linear contribution is interpreted here as being associated with bound charge. The conceptual simplicity of an element of total PV charge inducing simple fields, such as the geopotential being inversely dependent on distance from the anomaly, is then retained. The bound PV charge is then caused by the 'polarization' of the medium by the field due to the total charge. The non-linear bound charge contribution acts to modify the simple field only in the vicinity of the PV anomaly. In the far-field the non-linear bound charge contribution is negligible. This is why many of the essential characteristics associated with a PV anomaly are unchanged from the QG solution, e.g. a positive anomaly has cyclonic circulation with warm air above and cold air below as given in Fig. 1 of BT.

The linear superposition principle that applies in the quasi-geostrophic case does not carry over to the Ertel–Rossby case. However the non-linearity of the dielectric constant for PV is significant only in very close proximity to the anomaly. This means that apart from anomalies which are extremely close together linear superposition will be a rather accurate approximation. Somewhat different flow structures are superposed compared to those given by the quasi-geostrophic case so the inversion will be different mainly for this reason rather than due to non-linearity.

ACKNOWLEDGEMENTS

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APPENDIX

Following Shutts and Cullen (1987) one can introduce a new potential function, $P = \phi + f_0^2/2(x^2 + y^2)$, which has the property that $\zeta_{SG} = 1/f_0^3 \nabla P_x \times \nabla P_y$. Then the SPV, as defined in Eq. (22), can be written as the determinant of a matrix:

$$\mathrm{SPV} = \frac{\theta_0}{\rho_0 g f_0^3} \det \begin{bmatrix} P_{xx} & P_{xy} & P_{xz} \\ P_{yx} & P_{yy} & P_{yz} \\ P_{zx} & P_{zy} & P_{zz} \end{bmatrix}$$

Using a decomposition into a background mean state, denoted by an overbar, with uniform potential vorticity and a local deviation, denoted by a prime, then

$$P = \bar{P} + \phi' \quad \text{where} \quad \bar{P} = \bar{\phi} + \frac{f_0^2}{2}(x^2 + y^2)$$

Taking $\bar{\phi} = N_0^2 z^2/2 + gz$ gives $\overline{\mathrm{SPV}} = (\theta_0/\rho g f^3)\bar{P}_{xx}\bar{P}_{yy}\bar{P}_{zz} = \theta_0 f_0 N_0^2/\rho g$ and if the anomaly of semi-geostrophic PV is defined as before as $s_f' = (\mathrm{SPV} - \overline{\mathrm{SPV}}) f_0/\overline{\mathrm{SPV}}$ then the following can be derived:

$$s_f' = \frac{1}{f_0}\left(\phi'_{xx} + \phi'_{yy} + \frac{f_0^2}{N_0^2}\phi'_{zz}\right) + \frac{1}{f_0 N_0^2}((\phi'_{xx} + \phi'_{yy})\phi'_{zz} - \phi'^2_{xz} - \phi'^2_{yz})$$

$$+\frac{1}{f_0^3}(\phi'_{xx}\phi'_{yy}-\phi'^2_{xy})$$
$$+\frac{1}{f_0^3N_0^2}\det\begin{bmatrix}\phi'_{xx} & \phi'_{xy} & \phi'_{xz}\\ \phi'_{yx} & \phi'_{yy} & \phi'_{yz}\\ \phi'_{zx} & \phi'_{zy} & \phi'_{zz}\end{bmatrix}$$

Hence s'_f is the sum of the linear QG term, a quadratic term (that also appears in a geostrophic estimate of the PV), and one quadratic and one cubic term specifically associated with the SPV. In spherical polar coordinates, using r as defined in section 2, s'_f can be written with some algebraic manipulation as the sum of the three terms given in Eq. (23).

It is interesting to note that the SPV has a spherically symmetric form for a spherical PV anomaly field using the radial coordinate. (Note that the component terms are not all individually symmetrical. Also the mean state which has been subtracted from the geopotential is, apart from constant geopotential and potential temperature factors, also spherically symmetric in r; i.e. $\bar{P}=\phi_0+gz+(f_0^2/2)r^2$.) It is clear therefore that the Jacobian terms for the semi-geostrophic vorticity, although of small magnitude, are sufficient to make the Ertel–Rossby PV have a symmetric form.

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