A Microscopic Approach to Understanding the Moist Adiabatic Lapse Rate

First step: the vertical displacement

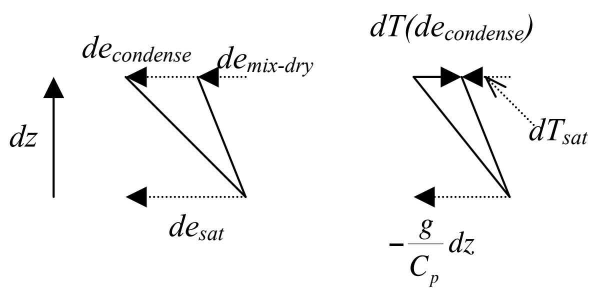
- We start with a saturated air parcel where $e = e_s$.
- The saturated air parcel is lifted upwards by a small vertical displacement
- This causes decompression because pressure decreases with altitude
- The actual vapor pressure of the air parcel, e, decreases with the vertical displacement because the mixing ratio is initially constant after the initial displacement and the pressure has decreased.
- The decompression also causes adiabatic expansion and cooling.
- The adiabatic decrease in temperature decreases the saturation vapor pressure, e_s.

Second step: the adjustment back to saturation

After the first step, e does not in general equal e_s. Therefore some adjustment occurs to bring the vapor pressure in the air parcel back into vapor pressure equilibrium and saturation. We can assume that $e > e_s$ after the initial vertical adiabatic displacement, then moving back to saturation equilibrium requires that some of the water vapor condense out, which

- reduces e
- warms the air parcel because of the temperature increase associated with the latent heat release
- the temperature increase (relative to the dry adiabat) increases e_s

until $e = e_s$ once again. This requires that the decrease in e caused by the condensation must equal the increase in e_s.



Before the displacement, $e_0 = e_{s0}$. After the vertical adiabatic displacement (Step 1),

- the mixing ratio of the water vapor in the air, e_0/p_0, has remained constant but the pressure has decreased fractionally an amount = $d\ln P/dz\ dz = -dz/H$. Therefore the change in water vapor associated with maintaining a constant mixing ratio is

$$de_{mix-dry} = -e_0 \frac{dz}{H} = -e_0 \frac{mgdz}{R^* T} \quad (1)$$

- The initial temperature is T_0. Assuming initially the temperature of the air parcel decreases according to the dry adiabat then the change of temperature, dT, of the displaced air parcel is

$$dT = -\frac{g}{C_p} dz$$

- The change in saturation vapor pressure at the new temperature is

$$de_{s0} = e_{s0} \frac{L}{R_v T^2} \frac{dT_s}{dz} dz \quad (2)$$

- In general $de_{s0} < de_{\text{mix-dry}}$ (note that both of these are negative). This means some water vapor will condense out and the temperature of the air will warm relative to the dry adiabat.

The solution is just enough water must condense out to warm the air just enough so that the air remains at saturation as it rises, no more, no less.

Water vapor

$$\underset{<0}{\delta e_{mix-dry}} + \underset{<0}{\delta e_{condense}} = \underset{<0}{\delta e_{sat}} \quad (3)$$

If the vertical displacement is upward, all of these changes are negative. We rearrange the equation to solve for the amount of water vapor that condenses out

$$\delta e_{condense} = \delta e_s - \delta e_{mix-dry} \quad (4)$$

Plugging in for the change in the saturation vapor pressure and the change due to the decrease in pressure with altitude while holding the water vapor mixing ratio constant

$$de_{condense} = \frac{e_s L_v}{R_v T^2} dT_s + e_0 \frac{mgdz}{R^* T} \quad (5)$$

Temperature

Next, we work through a similar sequence for the temperature

$$\underset{<0}{dT(de_s)} = \underset{<0}{dT_{ad-dry}} + \underset{>0}{dT(de_{condense})} \quad (6)$$

Note that the temperature change due to the condensation is positive. The condensation releases latent heat that causes the temperature of the surrounding air to increase. To determine the amount of latent heat released and the resulting temperature increase, we need to know the mass of water vapor condensed. This is done using the ideal gas law. We can use the ideal gas law for partial pressures

$$de = dn_v R^* T = d\rho_v R_v T \quad (7)$$

such that

$$d\rho_{v-condense} = \frac{de_{condense}}{R_v T} \quad (8)$$

Therefore the amount of latent heat released per unit volume is

$$L_v d\rho_{v-condense} = L_v \frac{de_{condense}}{R_v T} \quad (9)$$

This is transferred into the heat capacity of the air. The density of air is

$$\rho_{air} = \frac{P}{RT} \tag{10}$$

Therefore the energy transfer is

$$L_v \frac{de_{condense}}{R_v T} = \rho_{air} C_p dT_{condense} = \frac{P}{RT} C_p dT_{condense} \tag{11}$$

$$dT_{condense} = \frac{L_v}{C_p}\frac{R}{R_v}\frac{de_{condense}}{P} = \frac{L_v}{C_p}\frac{\mu_v}{\mu_d}\frac{de_{condense}}{P} \tag{12}$$

Now we need to be very careful of the sign of this term because we have defined $de_{condense}$ to be negative but the change in air temperature is positive. So we rewrite (6)

$$dT(de_s) \equiv dT_s = dT_{ad-dry} + dT(de_{condense}) \tag{6}$$

$$dT_s = -dz\frac{g}{C_p} - \frac{L_v}{C_p}\frac{\mu_v}{\mu_d}\frac{de_{condense}}{P} \tag{13}$$

$$de_{condense} = \frac{e_s L_v}{R_v T^2} dT_s + e_0 \frac{mgdz}{R^* T} \tag{5}$$

Now we plug (5) into (13)

$$\delta T_s = -\delta z\frac{g}{C_p} - \frac{L_v}{C_p}\frac{\mu_v}{\mu_d}\frac{1}{P}\left(\frac{e_s L_v}{R_v T^2}\delta T_s + e_0\frac{mg\delta z}{R^* T}\right) \tag{14}$$

$$\delta T_s\left[1 + \frac{L_v^2}{C_p R_v T^2}\frac{\mu_v}{\mu_d}\frac{e_s}{P}\right] = -\delta z\left[\frac{g}{C_p} + \frac{L_v}{C_p}\frac{\mu_v}{\mu_d}\frac{e_s}{P}\frac{\mu_d g}{R^* T}\right] = -\delta z\left[\frac{g}{C_p} + \frac{L_v}{C_p}\frac{e_s}{P}\frac{g}{R_v T}\right]$$

$$\delta T_s\left[1 + \frac{L_v^2}{C_p R_v T^2}\frac{\mu_v}{\mu_d}\frac{e_s}{P}\right] = -\delta z\frac{g}{C_p}\left[1 + \frac{e_s}{P}\frac{L_v}{R_v T}\right]$$

$$\frac{dT_s}{dz} = -\frac{g}{C_p}\frac{\left[1 + \frac{e_s}{P}\frac{L_v}{R_v T}\right]}{\left[1 + \frac{L_v^2}{C_p R_v T^2}\frac{\mu_v}{\mu_d}\frac{e_s}{P}\right]} \tag{15}$$

So the moist adiabatic lapse rate is a scaled version of the dry adiabatic lapse rate.

From the other derivation, we have

$$\Gamma_M = \Gamma_D \frac{1 + \frac{L_C q_s M_A}{R^* T}}{1 + \frac{L_C q_s}{c_p T}\frac{L_C M_W}{R^* T}} \tag{39}$$

These are the same.

Compare this to the definition form the ams glossary at
http://amsglossary.allenpress.com/glossary/search?id=moist-adiabatic-lapse-rate1

Note that $e_s = e_s(T)$ and $L_v = L_v(T)$ which makes integrating (15) versus altitude a bit tricky. We did not consider in (15) the subtle change in C_p as the amount of water changes or the heating of the condensed water when the vapor condenses.

The denominator term in parentheses can reach a value of 2 under very warm moist conditions such that the moist adiabat can reach an extreme value of ~ –0.5 g/C_p = -5 K/km.