Contents

1. Mathematical and Conceptual Tools

Irregular patterns are characteristic of turbulence, however, we can find a mean value and define an intensity of turbulence (as we saw in previous classes). Fortunately, the spectrum of turbulence shows a clear gap (*spectral gap*, that separates phenomena such as fronts and weather systems (on the order of 100 hours and 24 hours) from the microscale eddies (with duration between 10s and 10min). When we deal with the BL, the motions to the left are considered mean flow, while the motions to the right constitute turbulence. This is why we define the BL to respond to the surface with time scales of less than 1 hour.
In numerical models, eddies are not modeled directly but parameterized by stochastic approximations or models.

Figure 1: Figure 2.3 Spectrum of Wind

There is a net transfer of energy from larger to smaller eddies called the ”energy cascade”. At the smallest scale, the energy is dissipated into heat by molecular viscosity.

1a. *Mean and Turbulent Parts*

$$U = u' + \overline{U} \tag{1}$$

u' represents the flow that varies with periods less than one hour and can be above or below $\overline{U}$, which represents variations larger than one hour. In general we can have mean and turbulent parts for wind in 3D, moisture, pollutants and heat. $V = v' + \overline{V}$, $W = w' + \overline{W}$, $\theta_v = \theta_v' + \overline{\theta}_v$, $q = q' + \overline{q}$,$C = c' + \overline{C}$.

i. The Mean You can have a spatial, temporal or ensemble mean. In this class we will always refer to the temporal mean (unless otherwise stated.

$$\overline{A(s)} = \frac{1}{P}\int_{t=0}^{P} A(t,s)dt \tag{2}$$

$$\overline{A(s)} = \frac{1}{N}\sum_{i=1}^{N} A(i,s) \tag{3}$$

Properties of averaging:

$\overline{(A+B)} = \overline{A} + \overline{B}$

$\overline{c} = c$

$\overline{(cA)} = c\overline{A}$

$\overline{(\overline{A})} = \overline{A}$

$\overline{(\overline{A}B)} = \overline{A}\overline{B}$

$\overline{\left(\frac{dA}{dt}\right)} = \frac{d\bar{A}}{dt}$

In the last equation, notice: average of local slopes equals slope of the averages.

Leibniz' Theorem:

$$\frac{d}{dt}\left[\int_{S_1(t)}^{S_2(t)} A(t,s)ds\right] = \int_{S_1(t)}^{S_2(t)} \left[\frac{\partial A(t,s)}{dt}\right] ds + A(t,S_2)\frac{dS_2}{dt} - A(t,S_1)\frac{dS_1}{dt} \tag{4}$$

Notice that in this case, s refers to space. This theorem is important in BL studies because the upper boundary (the BL height) is variable in time.

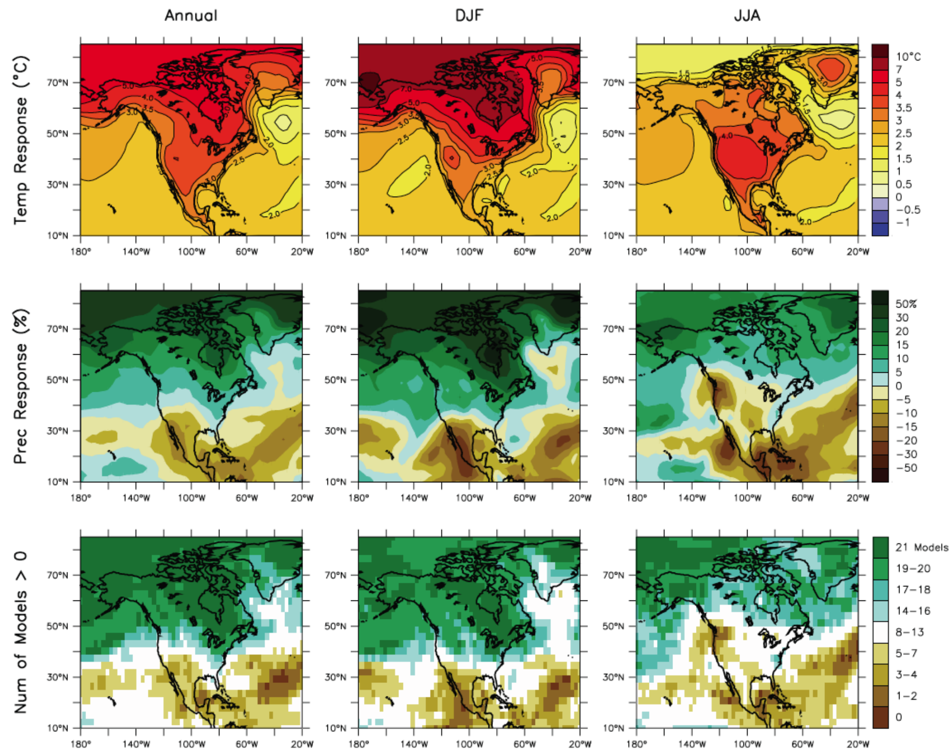

Figure 11.12. *Temperature and precipitation changes over North America from the MMD-A1B simulations. Top row: Annual mean, DJF and JJA temperature change between 1980 to 1999 and 2080 to 2099, averaged over 21 models. Middle row: same as top, but for fractional change in precipitation. Bottom row: number of models out of 21 that project increases in precipitation.*

Figure 2: The IPCC reports generally present ensemble means and temporal means.

Example:

Find the time rate of change of the average concentration a pollutant (denoted by c) within the BL, defined by integrating over the depth of the BL from z=0 to z=zi:

$$\frac{d}{dt}\left[\int_{z=0}^{z=zi} c(t,z)dz\right] = \int_0^{zi}\left[\frac{\partial c(t,z)}{dt}\right]dz + c(t,zi^+)\frac{dzi}{dt} \tag{5}$$

$$\frac{d}{dt}\left[\frac{zi}{zi}\int_{z=0}^{z=zi} c(t,z)dz\right] = \frac{zi}{zi}\int_0^{zi}\left[\frac{\partial c(t,z)}{dt}\right]dz + c(t,zi^+)\frac{dzi}{dt} \tag{6}$$

$$\frac{d}{dt}(zi\bar{c}) = zi\overline{\frac{dc}{dt}} + c(t, zi^{+})\frac{dzi}{dt} \tag{7}$$

Where the average denotes a spatial average in the vertical direction.

1b. *Reynolds Averaging*

Remember, a variable can be defined as the sum of its mean and deviations. We can then average...

$$\overline{A} = \overline{(\overline{A} + a')} = \overline{A} + \overline{a'} = \overline{A} \tag{8}$$

consequently, $\overline{a'} = 0$ as is expected when you are talking about deviations about a mean.

When multiplying:

$$\overline{(\overline{B}a')} = \bar{B}\bar{a}' = 0 \tag{9}$$

However, notice what happens when we multiply A and B:

$$\begin{aligned}
\overline{(AB)} &= \overline{(\overline{A} + a')(\overline{B} + b')} \\
&= \overline{(\bar{A}\bar{B} + a'\bar{B} + \bar{A}b' + a'b')} \\
&= \bar{A}\bar{B} + \overline{a'b'}
\end{aligned} \tag{10}$$

The nonlinear product $\overline{a'b'}$ is not necessarily zero. In the same manner, $\overline{a'a'}$, $\overline{a'b'^2}$, $\overline{a'^2b'^2}$ are not necessarily zero AND MUST BE RETAINED to properly model turbulence.

1c. *Variance, Standard Deviation and Turbulence Intensity*

Variance: Dispersion of data about the mean.

$$\sigma_a^2 = \frac{1}{N-1}\sum_{i=0}^{N-1}(A_i - \bar{A})^2 \tag{11}$$

This is a measure of dispersion of the population. When N is very large (as it is for most BL studies) $N \approx N-1$, and because $a' = A - \bar{A}$,

$$\sigma_a^2 = \overline{a'^2} \tag{12}$$

and this is true for winds and scalars.

When you see variables such as $\overline{u'^2}$, $\overline{q'^2}$, $\overline{\theta_v'^2}$, you can interpret as variances. As the mean wind increases, usually the intensity of turbulence increases.

1d. *Covariance and Correlation*

Using $N \approx N - 1$, we define the covariance as:

$$
\begin{aligned}
covar(A, B) &= \frac{1}{N}\sum_{i=0}^{N-1}(A_i - \bar{A})(B_i - \bar{B}) \\
&= \frac{1}{N}\sum_{i=0}^{N-1} a_i' b_i' \\
&= \overline{a'b'}
\end{aligned}
\tag{13}
$$

The nonlinear products have the same meaning as covariances. They indicate the degree of common relationship between variables.

For example, if A represents temperature and B represents vertical velocity.

$\overline{T'w'} > 0$ This is the most common case, lower 80% of the mixed layer.

$T' > 0$ and $w' > 0$: Warmer than average air rises (during the day when ground is warmer than air).

$T' < 0$ and $w' < 0$: Cooler than average air sinks because of higher density.

$\overline{T'w'} < 0$ Not very common

$T' < 0$ and $w' > 0$: Cool air rises due to mechanical turbulence in the early morning.

$T' > 0$ and $w' < 0$: Warm air sinks because of mechanical turbulence.

i. Linear Correlation Coefficient Indicates the normalized covariance:

$$
r_{AB} = \frac{\overline{a'b'}}{\sigma_a \sigma_b} \tag{14}
$$

Where the two variables are perfectly correlated if $r_{AB} = 1$ and perfectly negatively correlated when $r_{AB} = -1$

Figure 3: Fig2.7 Correlation Coefficient profiles in the convective mixed layer

Example: From the virtual potential temperature (K) and vertical wind velocity (m/s) data below, calculate the mean and standard deviation for each variable, the covariance and the linear correlation coefficient between variables.

Table 1: default

w	θ_v
0.5	295
-0.5	292
1	295
0.8	298
0.9	300

1e. *Summation (shorthand) Notation*

This will save us (a lot) of space and writing.

m, n and q are integer variable indices 1,2 or 3.
A_m is a velocity vector (U,V,W).
x_m is a component of distance (x,y,z).
δ_m is a unit vector (**i,j,k**).

If there are no free indices, the variable is a scalar.
One free index, a vector.
Two free indices, a tensor.

Kronecker Delta (scalar)

$$\delta_{mn} = \begin{cases} 1 & \text{for m=n,} \\ 0 & \text{for m} \neq \text{n} \end{cases}$$

Alternating Unit Tensor (scalar)

	w	theta	w'	theta'	w'^2	theta'^2	wtheta	w'theta'
0	0.5	295	-0.04	-1	0.0016	1	147.5	0.04
1	-0.5	292	-1.04	-4	1.0816	16	-146	4.16
2	1	295	0.46	-1	0.2116	1	295	-0.46
3	0.8	298	0.26	2	0.0676	4	238.4	0.52
4	0.9	300	0.36	4	0.1296	16	270	1.44
mean	0.54	296.00	0.00	0.00	0.30	7.60	160.98	1.14
avg(wtheta)	160.98							
avg(w)avg(theta)+avg(w'theta')	160.98							
biased standard deviation w	0.55							
biased standard deviation theta	2.76							
linear correlation coefficient	0.76							

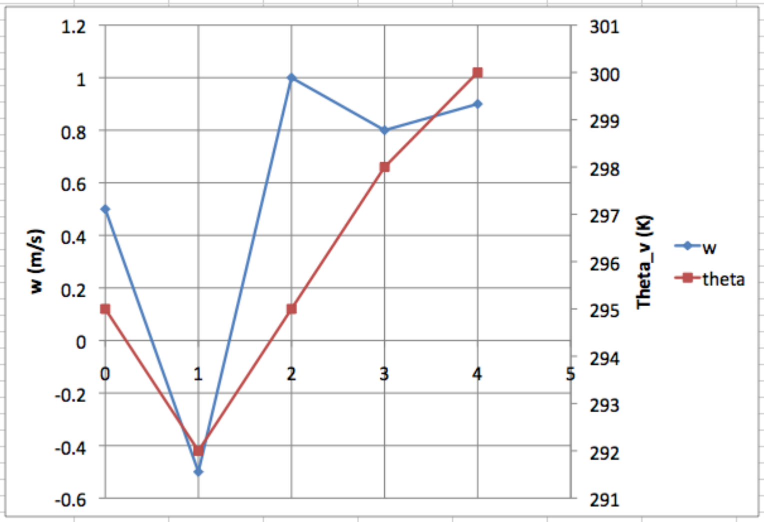

Figure 4: Example of calculation of variance, covariance, correlation coefficient

$$\epsilon_{mnq} = \begin{cases} +1 & \text{for mnq=123, 231 or 312} \\ -1 & \text{for mnq=321, 213 or 132} \\ 0 & \text{for any two or more indices alike.} \end{cases}$$

Rule 1 Whenever two identical indices appear in the same term, it is implied that there is a sum of that term over each value (1, 2, 3) of the repeated index.

Rule 2 Whenever one index appears unsummed (free) in a term, then that same index must appear unsummed in all terms in that equation. Hence, that equation effectively represents 3 equations for each value of the unsummed index.

Rule 3 The same index cannot appear more than twice in one term.

i. Example, equation of motion

$$\frac{\partial A_m}{\partial t} + B_n \frac{\partial A_m}{\partial X_n} = -\delta_{m3} g + f_c \epsilon_{mn3} B_n - \frac{1}{\rho}\frac{\partial p}{\partial X_m} + \frac{1}{\rho}\frac{\partial \tau_{mn}}{\partial X_n} \tag{15}$$

Rule 1 Sum over repeated indices.

$$\begin{aligned} \frac{\partial A_m}{\partial t} + B_1 \frac{\partial A_m}{\partial X_1} + B_2 \frac{\partial A_m}{\partial X_2} + B_3 \frac{\partial A_m}{\partial X_3} &= -\delta_{m3} g + f_c \epsilon_{m13} B_1 + f_c \epsilon_{m23} B_2 + 0 - \\ & \frac{1}{\rho}\frac{\partial p}{\partial X_m} + \frac{1}{\rho}\left(\frac{\partial \tau_{m1}}{\partial X_1} + \frac{\partial \tau_{m2}}{\partial X_2} + \frac{\partial \tau_{m3}}{\partial X_3}\right) \end{aligned} \tag{16}$$

Rule 2 Different equation for each free index.

$$\begin{aligned} \frac{\partial A_1}{\partial t} + B_1 \frac{\partial A_1}{\partial X_1} + B_2 \frac{\partial A_1}{\partial X_2} + B_3 \frac{\partial A_1}{\partial X_3} &= 0 + 0 + f_c B_2 + 0 - \\ & \frac{1}{\rho}\frac{\partial p}{\partial X_1} + \frac{1}{\rho}\left(\frac{\partial \tau_{11}}{\partial X_1} + \frac{\partial \tau_{12}}{\partial X_2} + \frac{\partial \tau_{13}}{\partial X_3}\right) \end{aligned} \tag{17}$$

$$\begin{aligned} \frac{\partial A_2}{\partial t} + B_1 \frac{\partial A_2}{\partial X_1} + B_2 \frac{\partial A_2}{\partial X_2} + B_3 \frac{\partial A_2}{\partial X_3} &= 0 - f_c B_1 + 0 + 0 - \\ & \frac{1}{\rho}\frac{\partial p}{\partial X_2} + \frac{1}{\rho}\left(\frac{\partial \tau_{21}}{\partial X_1} + \frac{\partial \tau_{22}}{\partial X_2} + \frac{\partial \tau_{23}}{\partial X_3}\right) \end{aligned} \tag{18}$$

$$\begin{aligned} \frac{\partial A_3}{\partial t} + B_1 \frac{\partial A_3}{\partial X_1} + B_2 \frac{\partial A_3}{\partial X_2} + B_3 \frac{\partial A_3}{\partial X_3} &= -g + 0 + 0 + 0 - \\ & \frac{1}{\rho}\frac{\partial p}{\partial X_3} + \frac{1}{\rho}\left(\frac{\partial \tau_{31}}{\partial X_1} + \frac{\partial \tau_{32}}{\partial X_2} + \frac{\partial \tau_{33}}{\partial X_3}\right) \end{aligned} \tag{19}$$

It is clear that using equation 20 is much more compact than using equations 17, 18 and 19. This is why we use Einstein's notation in this class. This equation represents the conservation of momentum equation, and will be discussed later...the more familiar way of expressing this equation is:

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\delta_{i3} g + f_c \epsilon_{ij3} U_j - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_j} \tag{20}$$

2. Conceptual Tools

2a. *Kinematic Flux*

Flux is the transfer of a quantity per unit area per unit time. You can have mass, heat, moisture, momentum and pollution fluxes (among others). Table 2 presents the fluxes in their regular and kinematic form.

Table 2: Flux and Kinematic Flux

Quantity	Flux		Kinematic Flux	
	Symbol	Units	Equation	Units
mass	$\tilde{M}$	$\frac{kg_{air}}{m^2 s}$	$M = \frac{\tilde{M}}{\rho_{air}}$	$\frac{m}{s}$
heat	$\tilde{Q}_H$	$\frac{J}{m^2 s}$	$Q_H = \frac{\tilde{Q}_H}{\rho_{air} C_P}$	$K\frac{m}{s}$
moisture	$\tilde{R}$	$\frac{kg_{water}}{m^2 s}$	$R = \frac{\tilde{R}}{\rho_{air}}$	$\frac{kg_{water}}{kg_{air}}\frac{m}{s}$
latent heat	$\tilde{Q}_E = L_v \tilde{R}$	$\frac{J}{m^2 s}$	$Q_E = \frac{\tilde{Q}_E}{\rho_{air} C_P}$	$K\frac{m}{s}$
momentum	$\tilde{F}$	$\frac{kg(m/s)}{m^2 s}$	$F = \frac{\tilde{F}}{\rho_{air}}$	$\frac{m}{s}\frac{m}{s}$
pollutant	$\tilde{X}$	$\frac{kg_{pollutant}}{m^2 s}$	$R = \frac{\tilde{X}}{\rho_{air}}$	$\frac{kg_{pollutant}}{kg_{air}}\frac{m}{s}$

where $L_v = 2.45^6$ J/kg at 20°C, $C_P = 1005$ j/(kg K)is the specific heat of air and $\rho_{air} = 1.12\ kg/m^3$ is the density of air, which is taken as a constant because the density change across it can be neglected.

The advantage of $kinematic fluxes$ is that we can easily measure them directly (wind speed, temperature, specific humidity, concentration) as opposed to heat, momentum etc. (for the regular fluxes). We can transform between regular and kinematic fluxes:

If we know $\tilde{Q}_H = 365W/m^2$ then we can calculate $Q_H = \tilde{Q}_H/(\rho_{air}C_P)$, so $Q_H = 365W/m^2/(1.12kg/m^3 * 1005J/(kgK)) = 0.3Km/s$.

Each of these fluxes has three components (one vertical and two horizontal). However, $momentum$ has an added dimension because the flux in any one direction can be the flux of U, V or W momentum. There are nine components of the flux to consider, so momentum is a second order tensor.

Fluxes can also be split into mean and turbulent parts:

Table 3: Mean and Turbulent Fluxes

Mean (Kinematic Advective KA)		Turbulent (Kinematic Eddy KE)	
Vertical KA heat flux	$\overline{W\theta}$	Vertical KE heat flux	$\overline{w'\theta'}$
Vertical KA moisture flux	$\overline{Wq}$	Vertical KE moisture flux	$\overline{w'q'}$
Vertical KA u momentum flux	$\overline{WU}$	Vertical KE u momentum flux	$\overline{w'u'}$
Zonal KA heat flux	$\overline{U\theta}$	Zonal KE heat flux	$\overline{u'\theta'}$

Example:
During the daytime, there is usually a superadiabatic profile close to the surface. Turbulent gusts bring warmer air up ($w' > 0, \theta' > 0$) and colder air down ($w' < 0, \theta' < 0$), so $\overline{w'\theta'} > 0$, even if $\overline{w'}$=0. *There is a net transport of heat* $\overline{w'\theta'} > 0$ *even though there is no net transport of mass* $\overline{w}' = 0$. As heat moves up, the lapse rate is more adiabatic.

At night, there is usually a subadiabatic profile close to the surface. Turbulent gusts bring warmer air down ($w' < 0, \theta' > 0$) and colder air up ($w' > 0, \theta' < 0$), so $\overline{w'\theta'} < 0$. As heat moves down, the lapse rate is more adiabatic.

While advective vertical fluxes are very small $\overline{W} \approx 0$, vertical turbulent fluxes are non-negligible. However, horizontal advective fluxes are quite large. [Show histograms].

2b. *Stress*

Stress is a force that tends to produce deformation in a body [force per unit area]. Pressure, Reynolds Stress and Viscous Shear Stress are the three types of stress

Figure 5: Figure 2.15, idealized soundings of turbulent fluxes in convective and stable BL.

that appear in the study of the atmosphere.

i. Pressure *Pressure acts on a fluid at rest.*
It acts equally in all directions (isotoropic). Isotropic pressure will cancel in all directions except in the direction normal to the object. Pressure is a scalar (doesn't depend on direction).
It tends to compress or expand objects.
At sea level, atmospheric pressure is 1.013×10^5 Pa (N/m^2). Kinematic pressure $\approx$ 82714 m^2/s^2...And while this is much larger than other stresses, it is almost always balanced by gravity.

ii. Reynolds Stress *Acts when fluid is in turbulent motion.*
When we derive the mean equations for the velocity components $\overline{u_i}$, new terms of the form $\partial\overline{(u'_i u'_j)}/\partial x_j$ arise. They are a result of the nonlinear terms that arise when we multiply velocities. The terms $\overline{(u'_i u'_j)}$ denote the turbulent momentum flux or *Reynolds stress*.
Physically we can think of this: the rate that air of different speeds is transported across any face of an object. The object tends to deform [identical to momentum flux].
Momentum flux in kinematic units $|\overline{u'w'}| = \tau_{Reynolds}$. For each cartesian direction we have three components, for a total of nine components. However, the Reynolds stress tensor is symmetric, so we only deal with 6 components.... $|\overline{u'w'}|$ = $|\overline{w'u'}|$.
It is important to emphasize that the Reynolds Stress is proportional to the *flow*, not the fluid.
Kinematic Reynolds Stress $\approx$ 0.05 m^2/s^2

iii. Viscous Stress *Acts when there are shearling motions on the fluid.*
Real fluid experiences tangential forces with a condition of no-slip at the boundary.
Intermolecular forces tend to drag the fluid adjacent to a moving portion.
Viscous stress depends on the *fluid*, so water exerts a greater viscous stress than air. Viscosity is the measure of these intermolecular forces.

A fluid, for which the viscous stress is linearly dependent on the shear is said to be a *Newtonian fluid*.
Velocity gradients are responsible for the rate of strain and deformation.
These gradients are perpendicular to the velocity.

Figure 6: Cartoon of Viscous Stress

In general, for two dimensions:

$$\tau_{ij} = \mu \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \tag{21}$$

τ_{ij} represents the tangential frictional force per unit area in N/m^2, where μ is the dynamic viscosity. We can put it in kinematic form by dividing by ρ ($\nu = \mu/\rho = 1.4607 \times 10^{-5}$ m^2/s)

$$\tau_{ij} = \nu \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \tag{22}$$

Kinematic viscous stress $\approx 7.304 \times 10^{-6}$ m^2/s^2.
We usually neglect viscous stress for mean wind *but NOT for turbulence*.

2c. *Friction Velocity*

At the surface the magnitude of the Reynolds stress is the total vertical flux of horizontal momentum measured at the surface:

$$\tau_{xz} = -\bar{\rho}\overline{u'w'_s} \tag{23}$$
$$\tau_{yz} = -\bar{\rho}\overline{v'w'_s} \tag{24}$$
$$\tag{25}$$

The total Reynolds stress is:

$$|\tau_{Reynolds}| = (\tau_{xz}^2 + \tau_{yz}^2)^{1/2} \tag{26}$$

We define a velocity scale called the *friction velocity* u_*, when turbulence is generated by wind shear, this is a very important scaling variable.

$$u_*^2 = |\tau_{Reynolds}|/\bar{\rho} = (\overline{u'w'_s}^2 + \overline{v'w'_s}^2)^{1/2} \tag{27}$$

We will also introduce the surface layer temperature scale

$$\theta_*^{SL} = \frac{-\overline{w'\theta'_s}}{u_*} \tag{28}$$

And the surface layer humidity scale

$$q_*^{SL} = \frac{-\overline{w'q'_s}}{u_*} \tag{29}$$