# Contents

# 1. Introduction to Governing Equations

The *Equations of motion* contain the time and space derivatives that require initial and boundary conditions for their solution.
Generally we don't forecast all eddy motions, we pick a cut-off eddy size and below this we include only the statistical effects of turbulence.
Mesoscale and synoptic models : cut-off is on the order of 10 to 100km.
Large Eddy simulation : cut-off is on the order of 100m.

However, averaging the equations leads to a situation where there are more unknowns than equations - the so-called closure problem. The closure problem leads us to consider the second-moment equations with particular attention to the turbulent kinetic energy (TKE) equation.
There is no analytical solution to the complete set of equations. So we must either simplify the equations and solve analytically, or use numerical models.

Remember, turbulent flows are rotational and three dimensional, they are dissipative so energy must be supplied to maintain the turbulence, fluid motions are not predictable in detail, the rates of transfer and mixing are orders of magnitude larger than the rate of molecular diffusion.

**1a.** *Methodology*

- Step 1. Identify the basic governing equations for boundary layer.
- Step 2. Expand dependent variables into mean and turbulent parts.
- Step 3. Apply Reynolds averaging to get mean variables within turbulent flow.
- Step 4. Obtain equations for turbulent departure from mean.
- Step 5. Obtain prognostic equations for turbulence statistics, like turbulence kinetic energy.

# 2. Equation of State

$$p = \rho R T_v \tag{1}$$

where $p$ is pressure, $\rho$ is the density of moist air, $T_v$ is the virtual absolute temperature and $R$ is the gas constant for dry air ($R = 287J/(kgK)$

**2a.** *Mean and Turbulent Parts*

$$\frac{\bar{p} + p'}{R} = (\bar{\rho} + \rho')(\overline{T_v} + T'_v) \tag{2}$$

$$\frac{\overline{p}}{R} + \frac{p'}{R} = \bar{\rho}\overline{T_v} + \bar{\rho}T'_v + \rho'\overline{T_v} + \rho'T'_v \tag{3}$$

**2b.** *Reynolds Averaging*

$$\frac{\overline{p}}{R} = \overline{\rho T_v} + \overline{\rho' T'_v} \tag{4}$$

because $\overline{\rho' T'_v} << \overline{\rho T_v}$

$$\boxed{\frac{\overline{p}}{R} = \overline{\rho T_v}} \tag{5}$$

Equation of state for mean variables

**2c.** *Equation for Turbulent Part*

If we subtract equation 5 from 3:

$$\frac{p'}{R} = \bar{\rho}T'_v + \rho'\overline{T_v} + \rho'T'_v \tag{6}$$

By dividing equation 6 by $\bar{p}/R = \overline{\rho T_v}$, and neglecting the term that looks like $\rho' T'_v/(\overline{\rho T_v})$, we obtain the *linearized perturbation ideal gas law*

$$\boxed{\frac{p'}{\bar{p}} = \frac{T'_v}{\overline{T_v}} + \frac{\rho'}{\bar{\rho}}} \tag{7}$$

Within the boundary layer, we can neglect the pressure perturbation over average pressure, and we can say that:

$$-\frac{T'_v}{\overline{T_v}} = \frac{\rho'}{\bar{\rho}} \tag{8}$$

Which is stating that warmer (colder) than average air is less dense (more dense) than average. This also allows us to substitute temperature fluctuations in place of density fluctuations.

# 3. Conservation of Mass

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho U_j)}{\partial x_j} = \frac{\partial \rho}{\partial t} + U_j \frac{\partial \rho}{\partial x_j} + \rho \frac{\partial U_j}{\partial x_j} = 0 \tag{9}$$

Because the definition of total derivative: $\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + U_i \frac{\partial \rho}{\partial x_i}$

$$\frac{d\rho}{dt} + \rho \frac{\partial U_j}{\partial x_j} = 0 \tag{10}$$

Within the boundary layer, $\frac{d\rho}{dt}/\rho << \frac{\partial U_j}{\partial x_j}$, so we use the incompressibility assumption:

$$\frac{\partial U_j}{\partial x_j} = 0 \tag{11}$$

**3a.** *Mean and Turbulent Parts*

$$\frac{\partial(\overline{U_j} + u'_j)}{\partial x_j} = 0 \tag{12}$$

**3b.** *Reynolds Averaging*

$$\boxed{\frac{\partial \overline{U_j}}{\partial x_j} = 0} \tag{13}$$

Conservation of mass for mean variables

**3c.** *Equation for Turbulent Part*

Subtracting equation 14 from 12:

$$\boxed{\frac{\partial u'_j}{\partial x_j} = 0} \tag{14}$$

Conservation of mass for turbulent variables

**3d.** *Flux form of advection terms*

The equations of motion have the advection term: $U_j \frac{\partial \zeta}{\partial x_j}$.
Using continuity equation, we know that $\zeta \frac{\partial U_j}{\partial x_j} = 0$. Then,

$$U_j \frac{\partial \zeta}{\partial x_j} = U_j \frac{\partial \zeta}{\partial x_j} + \zeta \frac{\partial U_j}{\partial x_j} = \frac{\partial (U_j \zeta)}{\partial x_j} \tag{15}$$

# 4. Conservation of Moisture

$$\frac{\partial q}{\partial t} + U_j \frac{\partial q}{\partial x_j} = \nu_q \frac{\partial^2 q}{\partial x_j^2} + \frac{S_q}{\rho_{air}} \tag{16}$$

Where $q$ is the specific humidity (mass of water per unit mass of moist air), $\nu_q$ is the molecular diffusivity for water vapor in the air, $S_q$ is moisture source term.

**4a.** *Mean and Turbulent Parts*

$$\frac{\partial \overline{q}}{\partial t} + \frac{\partial q'}{\partial t} + \overline{U_j} \frac{\partial \overline{q}}{\partial x_j} + \overline{U_j} \frac{\partial q'}{\partial x_j} + u_j' \frac{\partial \overline{q}}{\partial x_j} + u_j' \frac{\partial q'}{\partial x_j} = \nu_q \frac{\partial^2 \bar{q}}{\partial x_j^2} + \nu_q \frac{\partial^2 q'}{\partial x_j^2} + \frac{S_q}{\rho_{air}} \tag{17}$$

**4b.** *Reynols Averaging*

We perform Reynolds Averaging on 18 and express the turbulent advection term in flux form $\overline{u_j' \frac{\partial q'}{\partial x_j}} = \frac{\partial \overline{u_j' q'}}{\partial x_j}$

$$\boxed{\underbrace{\frac{\partial \overline{q}}{\partial t}}_{I} + \underbrace{\overline{U_j} \frac{\partial \overline{q}}{\partial x_j}}_{II} = \underbrace{\nu_q \frac{\partial^2 \bar{q}}{\partial x_j^2}}_{III} + \underbrace{\frac{S_q}{\rho_{air}}}_{IV} - \underbrace{\frac{\partial \overline{u_j' q'}}{\partial x_j}}_{V}} \tag{18}$$

Conservation equation for mean total moisture

- Term I represents storage of mean moisture
- Term II advection of mean moisture by mean wind

- Term III mean molecular diffussion of water vapor
- Term IV mean net body source term for additional moisture processes
- Term V Divergence of turbulent total moisture flux

**4c.** *Equation for Turbulent Part*

Subtracting Equation 18 from 18 :

$$\boxed{\frac{\partial q'}{\partial t} + \overline{U_j}\frac{\partial q'}{\partial x_j} + u'_j\frac{\partial \overline{q}}{\partial x_j} + u'_j\frac{\partial q'}{\partial x_j} = \nu_q\frac{\partial^2 q'}{\partial x_j^2} + \frac{\partial \overline{u'_j q'}}{\partial x_j}} \tag{19}$$

Prognostic equation for the perturbation part ($q'$).

# 5. Conservation of Scalar Quantity

$$\frac{\partial C}{\partial t} + U_j\frac{\partial C}{\partial x_j} = \nu_c\frac{\partial^2 C}{\partial x_j^2} + \frac{S_c}{\rho_{air}} \tag{20}$$

Where $C$ is the concentration of a tracer (mass of scalar per unit mass of moist air), $\nu_c$ is the molecular diffusivity for that scalar in the air, $S_c$ is net source term.

**5a.** *Mean and Turbulent Parts*

$$\frac{\partial \overline{C}}{\partial t} + \frac{\partial c'}{\partial t} + \overline{U_j}\frac{\partial \overline{C}}{\partial x_j} + \overline{U_j}\frac{\partial c'}{\partial x_j} + u'_j\frac{\partial \overline{C}}{\partial x_j} + u'_j\frac{\partial c'}{\partial x_j} = \nu_c\frac{\partial^2 \bar{C}}{\partial x_j^2} + \nu_c\frac{\partial^2 c'}{\partial x_j^2} + \frac{S_c}{\rho_{air}} \tag{21}$$

**5b.** *Reynols Averaging*

We perform Reynolds Averaging on 26 and express the turbulent advection term in flux form $\overline{u'_j\frac{\partial c'}{\partial x_j}} = \frac{\partial \overline{u'_j c'}}{\partial x_j}$

$$\boxed{\underbrace{\frac{\partial \overline{C}}{\partial t}}_{I} + \underbrace{\overline{U_j}\frac{\partial \overline{C}}{\partial x_j}}_{II} = \underbrace{\nu_c \frac{\partial^2 \overline{C}}{\partial x_j^2}}_{III} + \underbrace{\frac{S_c}{\rho_{air}}}_{IV} - \underbrace{\frac{\partial \overline{u'_j c'}}{\partial x_j}}_{V}} \tag{22}$$

Conservation equation for mean tracer C

- Term I represents storage of mean tracer C
- Term II advection of mean tracer by mean wind
- Term III mean molecular diffusion of tracer
- Term IV mean net body source term for additional tracer processes
- Term V Divergence of turbulent total tracer flux

**5c.** *Equation for Turbulent Part*

Subtracting Equation 26 from 26 :

$$\boxed{\frac{\partial c'}{\partial t} + \overline{U_j}\frac{\partial c'}{\partial x_j} + u'_j \frac{\partial \overline{C}}{\partial x_j} + u'_j \frac{\partial C'}{\partial x_j} = \nu_c \frac{\partial^2 c'}{\partial x_j^2} + \frac{\partial \overline{u'_j c'}}{\partial x_j}} \tag{23}$$

Prognostic equation for the perturbation part ($c'$).

# 6. Conservation of Heat

The First Law of Thermodynamics includes contributions from both sensible and latent heat. Remember that water vapor not only transports temperature, but also the potential to release or absorb additional latent heat during phase change.

$$\frac{\partial \theta}{\partial t} + U_j \frac{\partial \theta}{\partial x_j} = \nu_\theta \frac{\partial^2 \theta}{\partial x_j^2} - \frac{1}{\rho C_p}\frac{\partial Q_j *}{\partial x_j} - \frac{\partial L_p E}{\rho C_p} \tag{24}$$

Where $\nu_\theta$ is the thermal diffusivity, $L_p$ is the latent heat associated with the phase change of E ($L_v$=2.5$\times 10^6$ J/kg (gas-liquid),$L_f$=3.34$\times 10^5$ J/kg (solid-liquid), $L_s$=2.83$\times 10^6$ J/kg (gas-solid)).
$Q*_j$ is the component of net radiation in the $j^{th}$ direction, and $C_p$ is the specific

heat of moist air, related to the specific heat of dry air $C_{pd}$ = $1005 Jkg^{-1}K^{-1}$ by $C_p = C_{pd}(1 + 0.84q)$

- Term I storage term
- Term II advection term
- Term III mean molecular diffusion diffusion
- Term IV source term associated to radiation divergence
- Term V source term associated to latent heat released during phase change

**6a.** *Mean and Turbulent Parts*

$$\frac{\partial\overline{\theta}}{\partial t} + \frac{\partial\theta'}{\partial t} + \overline{U_j}\frac{\partial\overline{\theta}}{\partial x_j} + \overline{U_j}\frac{\partial\theta'}{\partial x_j} + u'_j\frac{\partial\overline{\theta}}{\partial x_j} + u'_j\frac{\partial\theta'}{\partial x_j} = \nu_\theta\frac{\partial^2\overline{\theta}}{\partial x_j^2} + \nu_\theta\frac{\partial^2\theta'}{\partial x_j^2} - \frac{1}{\overline{\rho}C_p}\frac{\partial\overline{Q_j*}}{\partial x_j} - \frac{1}{\overline{\rho}C_p}\frac{\partial Q_j*'}{\partial x_j} - \frac{L_vE}{\overline{\rho}C_p} \tag{25}$$

**6b.** *Reynols Averaging*

We perform Reynolds Averaging on 26 and express the turbulent advection term in flux form $\overline{u'_j\frac{\partial\theta'}{\partial x_j}} = \frac{\partial\overline{u'_j\theta'}}{\partial x_j}$

$$\boxed{\underbrace{\frac{\partial\overline{\theta}}{\partial t}}_{I} + \underbrace{\overline{U_j}\frac{\partial\overline{\theta}}{\partial x_j}}_{II} = \underbrace{\nu_\theta\frac{\partial^2\overline{\theta}}{\partial x_j^2}}_{III} - \underbrace{\frac{1}{\overline{\rho}C_p}\frac{\partial\overline{Q_j*}}{\partial x_j}}_{IV} - \underbrace{\frac{L_vE}{\overline{\rho}C_p}}_{V} - \underbrace{\frac{\partial\overline{u'_j\theta'}}{\partial x_j}}_{VI}} \tag{26}$$

Conservation equation for heat

- Term I represents storage of heat
- Term II advection of heat by mean wind

- Term III mean molecular conduction of heat
- Term IV mean net body source associated with radiation divergence
- Term V mean net body source associated with latent heat release
- Term VI divergence of turbulent heat flux

**6c.** *Equation for Turbulent Part*

Subtracting Equation 26 from 26 :

$$\boxed{\frac{\partial \theta'}{\partial t} + \overline{U_j}\frac{\partial \theta'}{\partial x_j} + u_j'\frac{\partial \overline{\theta}}{\partial x_j} + u_j'\frac{\partial \theta'}{\partial x_j} = \nu_c \frac{\partial^2 \theta'}{\partial x_j^2} + \frac{\partial \overline{u_j'\theta'}}{\partial x_j} - \frac{1}{\overline{\rho} C_p}\frac{\partial Q_j'*}{\partial x_j}} \tag{27}$$

Prognostic equation for the perturbation part ($\theta'$).

# 7. Conservation of Momentum

When discussing Einstein's (summation) notation, we used the example of the equation of conservation of momentum:

$$\underbrace{\frac{\partial U_i}{\partial t}}_{I} + \underbrace{U_j \frac{\partial U_i}{\partial x_j}}_{II} = \underbrace{-\delta_{i3} g}_{III} + \underbrace{f_c \epsilon_{ij3} U_j}_{IV} - \underbrace{\frac{1}{\rho}\frac{\partial p}{\partial x_i}}_{V} + \underbrace{\frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_j}}_{VI} \tag{28}$$

- Term I represents storage of momentum (inertia).
- Term II advection
- Term III vertical effect of gravity
- Term IV Coriolis effect where $f_c = 1.45 \times 10^-4$ sin $\phi$ ($\phi$ is latitude)
- Term V pressure gradient forces
- Term VI viscous stress

Remember that the viscous stress in Term VI can be expressed as: $\tau_{ij} = \mu\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)$, so

$$\frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_j} = \frac{\mu}{\rho}\frac{\partial}{\partial x_j}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) = \nu\frac{\partial}{\partial x_j}\frac{\partial U_i}{\partial x_j} + \nu\frac{\partial}{\partial x_i}\frac{\partial U_j}{\partial x_j} \tag{29}$$

Because of the conservation of mass $\frac{\partial U_j}{\partial x_j} = 0$

$$\text{Term VI} = \nu\frac{\partial^2 U_i}{\partial x_j^2} \tag{30}$$

$$\underbrace{\frac{\partial U_i}{\partial t}}_{I} + \underbrace{U_j\frac{\partial U_i}{\partial x_j}}_{II} = \underbrace{-\delta_{i3}g}_{III} + \underbrace{f_c\epsilon_{ij3}U_j}_{IV} - \underbrace{\frac{1}{\rho}\frac{\partial p}{\partial x_i}}_{V} + \underbrace{\nu\frac{\partial^2 U_i}{\partial x_j^2}}_{VI} \tag{31}$$

**7a.** *Mean and Turbulent Parts*

I will use Boussinesq approximation (explained in pages 83 and 84). This approximation neglects density variations in the storage term but retains them in the gravity term. This is essentially what will drive buoyancy in the equations of motion, and when applied to the vertical momentum equation it ensures that warmer than average air is accelerated upward.
*in practical terms we will replace every occurrence of $\rho$ with $\bar{\rho}$ and replace every occurrence of $g$ with $\left(g - (\theta_v'/\bar{\theta}_v)g\right)$*

$$\begin{aligned}\frac{\partial(\overline{U_i} + u_i')}{\partial t} + (\overline{U_j} + u_i j)\frac{\partial(\overline{U_i} + u_i')}{\partial x_j} \quad = \\ -\delta_{i3}\left(g - \frac{\theta_v'}{\bar{\theta}_v}g\right) + \\ f_c\epsilon_{ij3}(\overline{U_j} + u_j') - \frac{1}{\rho}\frac{\partial(\overline{p} + p')}{\partial x_i} + \nu\frac{\partial^2(\overline{U_i} + u_i')}{\partial x_j^2}\end{aligned} \tag{32}$$

We expand the above equation

$$\frac{\partial \overline{U_i}}{\partial t} + \frac{\partial u_i'}{\partial t} + \overline{U_j}\frac{\partial \overline{U_i}}{\partial xj} + \overline{U_j}\frac{\partial u_i'}{\partial xj} + u_j'\frac{\partial \overline{U_i}}{\partial xj} + u_j'\frac{\partial u_i'}{\partial xj} \quad = \quad -\delta_{i3}g - \delta_{i3}\frac{\theta_v'}{\overline{\theta_v}}g + f_c\epsilon_{ij3}\overline{U_j} + f_c\epsilon_{ij3}u_j' - \frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} - \frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \nu\frac{\partial^2 \overline{U_i}}{\partial x_j^2} + \nu\frac{\partial^2 u_i'}{\partial x_j^2} \tag{33}$$

**7b.** *Reynolds Averaging*

We do Reynolds Averaging and express the turbulent advection term in flux form $\overline{u_i'\frac{\partial u_j'}{\partial x_j}} = \frac{\partial \overline{u_j' u_i'}}{\partial x_j}$

$$\boxed{\underbrace{\frac{\partial \overline{U_i}}{\partial t}}_{I} + \underbrace{\overline{U_j}\frac{\partial \overline{U_i}}{\partial x_j}}_{II} = \underbrace{-\delta_{i3}g}_{III} + \underbrace{f_c\epsilon_{ij3}\overline{U_j}}_{IV} - \underbrace{\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i}}_{V} + \underbrace{\nu\frac{\partial^2 \overline{U_i}}{\partial x_j^2}}_{VI} - \underbrace{\frac{\partial \overline{u_j' u_i'}}{\partial x_j}}_{VII}} \tag{34}$$

Conservation of momentum for mean variables. Forecast equation for mean wind.

- Term I represents storage of mean momentum (inertia).
- Term II advection of mean momentum by mean wind
- Term III vertical effect of gravity
- Term IV Coriolis effect
- Term V mean pressure gradient forces
- Term VI viscous stress on mean motions
- Term VII Influence of Reynold's stress on mean motions. Also described as divergence of turbulent momentum flux.

Notice that this means that **turbulence must be considered in making forecasts in the turbulent boundary layer even if we are trying to forecast mean quantities**

**7c.** *Equation for Turbulent Part*

If we subtract equation 34 from 34, we get the following equation:

$$\boxed{\frac{\partial u_i'}{\partial t} + \overline{U_j}\frac{\partial u_i'}{\partial x_j} + u_j'\frac{\partial \overline{U_i}}{\partial x_j} + u_j'\frac{\partial u_i'}{\partial x_j} = \delta_{i3}\frac{\theta_v'}{\overline{\theta_v}}g + f_c\epsilon_{ij3}u_j' - \frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \nu\frac{\partial^2 u_i'}{\partial x_j^2} + \frac{\partial \overline{u_j'u_i'}}{\partial x_j}} \tag{35}$$

Conservation of momentum for turbulent variables. Forecast equation for turbulent gusts.

# 8. Boussinesq Approximation

Let's look at the momentum equation in the vertical direction (i=3):

$$\frac{dW}{dt} = -g - \frac{1}{\rho}\frac{\partial p}{\partial z} + \nu \frac{\partial^2 W}{\partial z^2} \tag{36}$$

We can multiply by $\rho$ and expand into mean and turbulent parts:

$$(\overline{\rho} + \rho')\frac{d(\overline{W} + w')}{dt} = -(\overline{\rho} + \rho')g - \frac{\partial(\overline{P} + p')}{\partial z} + \mu \frac{\partial^2(\overline{W} + w')}{\partial z^2} \tag{37}$$

Dividing by $\overline{\rho}$ and rearranging:
We can multiply by $\rho$ and expand into mean and turbulent parts:

$$(1 + \rho'/\overline{\rho})\frac{d(\overline{W} + w')}{dt} = -(\rho'/\overline{\rho})g - \frac{1}{\overline{\rho}}\frac{\partial p'}{\partial z} + \nu \frac{\partial^2(\overline{W} + w')}{\partial z^2} - \frac{1}{\overline{\rho}}\left(\frac{\partial \overline{P}}{\partial z} + \overline{\rho} g\right) \tag{38}$$

1. Assuming the mean state is in hydrostatic equilibrium, the last term = 0.
2. $(1 + \rho'/\overline{\rho}) \approx 1$ in the storage term.
3. $(\rho'/\overline{\rho})g$ is NOT negligible - IT MUST STAY!
4. Only for the momentum equation we can neglect subsidence $\overline{W}$ because it is always paired with the term $w'$ which is larger

$$\frac{dw'}{dt} = -(\rho'/\overline{\rho})g - \frac{1}{\overline{\rho}}\frac{\partial p'}{\partial z} + \nu \frac{\partial^2(w')}{\partial z^2} \tag{39}$$

The process of neglecting density variations in the storage term, but retaining them in the gravity term is called the **Boussinesq Approximation**. The practical application of Boussinesq Approximation: Given any of the original governing equations, replace every occurrence of $\rho$ with $\overline{\rho}$ and every occurrence of g with $[g + (\rho'/\overline{\rho})g] \approx [g - (\theta'_v/\overline{\theta_v})g]$

Notice the physical meaning of equation 39: the first two terms indicate that warmer than average air is accelerated upward, while the last two terms describe the effects of pressure and viscous stress on that motion.