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1. Universal Derivation for Variance terms

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3. Reynolds average
4. The terms that look like $u'_j\frac{\partial\zeta'^2}{\partial x_j}$ can be turned into flux form by using the continuity equation multiplied by ζ'^2 (which is equal to zero). $u'_j\frac{\partial\zeta'^2}{\partial x_j} + \zeta'^2\frac{\partial u'_j}{\partial x_j} = \frac{\partial\zeta'^2 u'_j}{\partial x_j}$

1a. *Example: Momentum Variance*

If we take equation **??**, and multiply by $2u'_i$, we obtain:

$$2u'_i\frac{\partial u'_i}{\partial t} \quad + \quad 2\overline{U_j}u'_i\frac{\partial u'_i}{\partial x_j} + 2u'_iu'_j\frac{\partial\overline{U_i}}{\partial x_j} + 2u'_ju'_i\frac{\partial u'_i}{\partial x_j} = \tag{1}$$

$$-2u'_i\delta_{i3}\frac{\theta'_v}{\overline{\theta_v}}g + 2f_c\epsilon_{ij3}u'_iu'_j - \frac{1}{\rho}\frac{\partial p'}{\partial x_i} + 2\nu u'_i\frac{\partial^2u'_i}{\partial x_j^2} + 2u'_i\frac{\partial\overline{u'_ju'_i}}{\partial x_j} \tag{2}$$

Then, the terms that look like $2u'_i\frac{\partial u'_i}{\partial t}$ will be expressed as $\frac{\partial u'^2_i}{\partial t}$

$$\frac{\partial u'^2_i}{\partial t} \quad + \quad \overline{U_j}\frac{\partial u'^2_i}{\partial x_j} + 2u'_iu'_j\frac{\partial\overline{U_i}}{\partial x_j} + u'_j\frac{\partial u'^2_i}{\partial x_j} = \tag{3}$$

$$-2u'_i\delta_{i3}\frac{\theta'_v}{\theta_v}g + 2f_c\epsilon_{ij3}u'_iu'_j - 2\frac{u'_i}{\rho}\frac{\partial p'}{\partial x_i} + 2\nu u'_i\frac{\partial^2u'_i}{\partial x_j^2} + 2u'_i\frac{\partial\overline{u'_ju'_i}}{\partial x_j} \tag{4}$$

1b. *Reynolds Averaging*

$$\frac{\partial \overline{u_i'^2}}{\partial t} \quad + \quad \overline{U_j}\frac{\partial \overline{u_i'^2}}{\partial x_j} + 2\overline{u_i' u_j'}\frac{\partial \overline{U_i}}{\partial x_j} + \overline{u_j' \frac{\partial u_i'^2}{\partial x_j}} = \tag{5}$$

$$-2\delta_{i3}\overline{u_i'\frac{\theta_v'}{\overline{\theta_v}}}g + 2f_c\epsilon_{ij3}\overline{u_i' u_j'} - 2\overline{\frac{u_i'}{\rho}\frac{\partial p'}{\partial x_i}} + 2\nu \overline{u_i'\frac{\partial^2 u_i'}{\partial x_j^2}} + 2\overline{u_i'\frac{\partial \overline{u_j' u_i'}}{\partial x_j}} \tag{6}$$

Now we perform several simplifications:

- Last term is 0 because $\overline{u_i'} = 0$
- Last term on left hand side can be turned into flux form by adding $u_i^2 \frac{\partial u_j'}{\partial x_j}$ which is equal to zero by continuity.
- Simplifying the dissipation term to $-2\nu\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2}$ = -2ϵ(see pg 122 in book for derivation).
- Put the pressure perturbation term in flux form $-\left(\frac{2}{\bar{\rho}}\right)\frac{\partial(\overline{u_i' p'})}{\partial x_i}$ using the fact that the turbulence continuity equation is zero.
- Coriolis is zero for velocity variances (try it!)

$$\frac{\partial \overline{u_i'^2}}{\partial t} + \overline{U_j}\frac{\partial \overline{u_i'^2}}{\partial x_j} = 2\delta_{i3}\frac{\overline{u_i'\theta_v'}}{\overline{\theta_v}}g - 2\overline{u_i' u_j'}\frac{\partial \overline{U_i}}{\partial x_j} + \frac{\overline{\partial u_j' u_i'^2}}{\partial x_j} - \left(\frac{2}{\bar{\rho}}\right)\frac{\partial(\overline{u_i' p'})}{\partial x_i} - 2\epsilon \tag{7}$$

The same is done for the prognostic equations for moisture variance q'^2, potential temperature variance θ'^2, variance of a scalar quantity c'^2.

2. Universal Derivation for Turbulent Fluxes

Equations for the mean variables in turbulent flow contain divergence terms of turbulent fluxes $\overline{u_i'u_j'}$, $\overline{\theta' u_j'}$, $\overline{q' u_j'}$. We can find prognostic equations for these fluxes.

1. Multiply perturbation equation for ζ' by u_i and Reynolds average.

2. Multiply the momentum perturbation equation (u_i') by ζ'

3. Add the two resulting equations

4. Use the continuity equation to get the turbulent transport terms into flux form-merge any other terms

2a. *Example: Moisture Flux*

Start with the momentum perturbation equation and multiply by moisture perturbation, Reynolds average.

$$\overline{q'\frac{\partial u_i'}{\partial t}}+\overline{U_j}\overline{q'\frac{\partial u_i'}{\partial x_j}}+\overline{q'u_j'}\frac{\partial \overline{U_i}}{\partial x_j}+\overline{q'u_j'\frac{\partial u_i'}{\partial x_j}} = \overline{q'\delta_{i3}\frac{\theta_v'}{\overline{\theta_v}}}g+f_c\epsilon_{ij3}\overline{q'u_j'}-\overline{\frac{q'}{\rho}\frac{\partial p'}{\partial x_i}}+\nu\overline{q'\frac{\partial^2 u_i'}{\partial x_j^2}}+\overline{q'\frac{\partial \overline{u_j'u_i'}}{\partial x_j}} \tag{8}$$

(last term is equal to zero when Reynolds averaging.)

Similarly, we start with the moisture perturbation equation and multiply by u_i' and Reynolds average.

$$\overline{u'\frac{\partial q'}{\partial t}}+\overline{U_j}\overline{u'\frac{\partial q'}{\partial x_j}}+\overline{u_i'u_j'}\frac{\partial \overline{q}}{\partial x_j}+\overline{u_i'u_j'\frac{\partial q'}{\partial x_j}} = \nu_q\overline{u_i'\frac{\partial^2 q'}{\partial x_j^2}}+\overline{u_i'\frac{\partial \overline{u_j'q'}}{\partial x_j}} \tag{9}$$

(last term is equal to zero when Reynolds averaging.) Now we add the equations 8 and 13, and put the turbulent flux divergence terms into flux form. We also assume $\nu = \nu_q$.

$$\frac{\partial \overline{u_i'q'}}{\partial t} \quad + \quad \overline{U_j}\frac{\partial \overline{u_i'q'}}{\partial x_j}+\overline{u_i'u_j'}\frac{\partial \overline{q}}{\partial x_j}+\overline{q'u_j'}\frac{\partial \overline{U_i}}{\partial x_j}+\frac{\partial \overline{(q'u_i'u_j')}}{\partial x_j} = \tag{10}$$

$$\delta_{i3}\frac{\overline{q'\theta_v'}}{\overline{\theta_v}}g+f_c\epsilon_{ij3}\overline{q'u_j'}-\frac{1}{\overline{\rho}}\left(\frac{\partial(\overline{q'p'})}{\partial x_i}-\overline{p'\frac{\partial q'}{\partial x_i}}\right)+\frac{\nu\partial^2\overline{(q'u_i')}}{\partial x_j^2}-2\nu\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)\left(\frac{\partial q'}{\partial x_j}\right)} \tag{11}$$

neglect Coriolis, pressure diffusion, molecular diffusion of turublent fluxes, and substitute the last term for $2\epsilon_{u_i q}$

$$\frac{\partial \overline{u_i' q'}}{\partial t} + \overline{U_j}\frac{\partial \overline{u_i' q'}}{\partial x_j} + \overline{u_i' u_j'}\frac{\partial \overline{q}}{\partial x_j} + \overline{q' u_j'}\frac{\partial \overline{U_i}}{\partial x_j} + \frac{\partial \overline{(q' u_i' u_j')}}{\partial x_j} = \tag{12}$$

$$\delta_{i3}\frac{\overline{q'\theta_v'}}{\overline{\theta_v}}g + \frac{1}{\rho}\left(\overline{p'\frac{\partial q'}{\partial x_i}}\right) - 2\epsilon_{u_i q} \tag{13}$$

First term is the storage term, second term is advection term, third, fourth and sixth are production/consumption terms, fifth is a turbulent transport, term seven is a redistribution and term eight is a molecular destruction (dissipation) of turbulent moisture flux).

3. Turbulence Kinetic Energy

Kinetic energy is defined as $KE = \frac{1}{2}mM^2$, where m is mass and M is velocity. We can also use kinetic energy per unit mass $KE/m = \frac{1}{2}M^2$. We can divide into mean and turbulence parts.

$$MKE/m = \frac{1}{2}(\bar{U}^2 + \bar{V}^2 + \bar{W}^2) \tag{14}$$

$$e = \frac{1}{2}(u'^2 + v'^2 + w'^2) = \frac{1}{2}u_i^2 \tag{15}$$

Where e is instantaneous, we can average over the instantaneous flows to obtain the turbulence kinetic energy (TKE) per unit mass:

$$TKE/m = \frac{1}{2}(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}) \tag{16}$$

Look how the *variance* will play a very important role in defining TKE. TKE is one of the most important quantities used to study the turbulent BL. TKE will tell us whether the BL will become more turbulent or if turbulence will decay.

- Turbulence Production: Buoyant thermals and mechanical eddies.
- Turbulence Suppression: statically stable lapse rate, and dissipated into heat by molecular viscosity.

There is a clear diurnal cycle of TKE in convective conditions:

Figure 1: Diurnal variability of TKE and different TKE profiles for different static stability conditions.

We can multiply by 0.5 and obtain the equation for TKE:

$$\boxed{\underbrace{\frac{\partial \overline{e}}{\partial t}}_{I} + \underbrace{\overline{U_j}\frac{\partial \overline{e}}{\partial x_j}}_{II} = \underbrace{\delta_{i3}\frac{\overline{u_i'\theta_v'}}{\overline{\theta_v}}g}_{III} - \underbrace{\overline{u_i'u_j'}\frac{\partial \overline{U_i}}{\partial x_j}}_{IV} - \underbrace{\frac{\partial \overline{u_j'e}}{\partial x_j}}_{V} - \underbrace{\left(\frac{1}{\bar{\rho}}\right)\frac{\partial(\overline{u_i'p'})}{\partial x_i}}_{VI} - \underbrace{\epsilon}_{VII}} \tag{17}$$

- Term I represents local storage of TKE. TKE storage increases from early morning with a peak in the early afternoon. $\approx 5 \times 10^{-5} m^3/s^3$. Over oceans, the storage term can be neglected (small diurnal cycle), which means that the intensity of turbulence doesn't change significantly with time.

- Term II advection of TKE by mean wind. Many times is neglected assuming horizontal homogeneity, however, this might not be valid for heterogeneous terrain (think of a reservoir).

- Term III *buoyant production or consumption term*. It can production term if the heat flux $\overline{u_i'\theta_v'}$ is positive (associated to *thermals* over land during day) or loss term when negative (during land during night). Depends on the flux of virtual potential temperature. This term is very important for days of free convection and can be used to normalize the TKE equation, at the surface Term III=w_*^3/z_i. Only acts in the vertical (anisotropic). Static stability tends to suppress or consume TKE, and is associated with negative Term III, and this occurs when the surface is colder than the overlying air.

- Term IV *mechanical production/loss term*. Associated with mean wind shear in the presence of Momentum flux is in opposite sign of wind shear, so their multiplication has a negative sign. Largest at the surface, because of the large wind shear, however, you can have large wind shear at the top of the ML with geostrophic winds above. Greatest contribution on windy days, during synoptic cyclones. Except in thunderstorms, W shear is negligible in the BL. Produces turbulence primarily in the horizontal (anisotropic).

- Term V is the turbulent transport of TKE, how TKE is moved through turbulent eddies. It is a flux divergence term - if integrated throughout the BL, it is zero, so it only redistributes turbulence (not a production/loss term). Maximum vertical transport at $z/z_i = 0.3$. Vertical transport dominates in the middle, horizontal dominates near the surface.

- Term VI describes how TKE is redistributed by pressure perturbations, often associated to oscillations (buoyancy or gravity waves). Very small pressure perturbations, cannot really be measured, so this term is usually calculated as a residual. This term can redistribute TKE but can also drain energy out of the BL.

- Term VII viscous dissipation of TKE, so the conversion of TKE into heat. Molecular destruction is greatest for the smallest eddy sizes, so intense small-scale turbulence induces larger dissipation. Largest near the surface, and then become constant, rapidly decrease to zero above the ML. At night TKE and dissipation vary rapidly with height. Generally, the greatest dissipation will occur where there is largest production - however there is not a perfect balance.

We can simplify if we align the system with the mean wind, assume horizontal homogeneity and neglect subsidence (notice that all the partial derivatives in x and y are zero because of horizontal homogeneity).

$$\frac{\partial \overline{e}}{\partial t} = \frac{\overline{w'\theta_v'}}{\overline{\theta_v}} g - \overline{w'u'}\frac{\partial \overline{U}}{\partial z} - \frac{\overline{\partial w'e}}{\partial z} - \left(\frac{1}{\bar{\rho}}\right)\frac{\partial(\overline{w'p'})}{\partial z} - \epsilon \tag{18}$$

3a. *Transfer of Energy and Mean Kinetic Energy*

Shear production and buoyant production terms are large and positive for large eddy sizes. There is a cascade of energy away from the large eddies towards the small eddies. At small eddy sizes, the production is close to zero and the dissipation is very large. This can be thought of as an inertial process where large eddies bump into smaller ones and transfer their inertia, the middle portion of the spectrum is the inertial subrange.

Figure 2: Figure 5.16

We can also evaluate the prognostic equation for mean kinetic energy (MKE). To do this, we multiply the equation for mean wind by $\overline{U_i}$. When you do, you get a term that looks like $\overline{u_i'u_j'}\frac{\partial \overline{U_i}}{\partial x_j}$, which is the same term as in the TKE equation, however they have opposite sign.

$$\frac{\partial MKE}{\partial t} = ... + \overline{u_i'u_j'}\frac{\partial \overline{U_i}}{\partial x_j} \tag{19}$$

$$\frac{\partial TKE}{\partial t} = ... - \overline{u_i'u_j'}\frac{\partial \overline{U_i}}{\partial x_j} \tag{20}$$

This is telling us that the energy that is mechanically produced as turbulence is lost from the mean flow.