

# 1. Water Vapor in Air

Water appears in all three phases in the earth's atmosphere - solid, liquid and vapor - and it is one of the most important components, not only because it is essential to life, but because the release of latent heat of vaporization when vapor condenses into the liquid or solid phases, together with the earth's rotation, drives the large-scale circulation of the atmosphere. Here, we show various measures of the variable atmospheric water vapor content.

## 1) VAPOR PRESSURE $e$

To a good approximation, water vapor behaves as a perfect gas. Thus, its equation of state is:

$$e = \rho_v R_v T \quad (1)$$

$$e \alpha_v = R_v T \quad (2)$$

$$e \alpha_v = \frac{R_d T}{\epsilon} \quad (3)$$

$e$  is vapor pressure  $R_v = R^*/M_v = 461.5 \text{ J kg}^{-1} \text{ K}^{-1}$  and  $M_v = 18.01 \text{ g mol}^{-1}$ ,  $\epsilon = M_v/M_d = 0.622$ . The vapor pressure is the partial pressure of the water vapor.

## 2) SATURATION VAPOR PRESSURE $e_s$

The saturation vapor pressure is the partial pressure of the water vapor in equilibrium with a plane surface of pure water. That means that the rate of condensation is equal to the rate of evaporation.  $e_s$  is a function of temperature alone, *it doesn't depend on the vapor content of the air*.  $e_s(T)$  is highly nonlinear as it increases rapidly with increasing temperature. This explains why the amount of atmospheric water vapor will likely increase with global warming caused by increasing concentration of greenhouse gases. This pressure is defined by the integral of the Clausius Clapeyron equation [you will do this in a HW]. There are also approximate expressions for  $e_s$ , one of the most used is:

$$e_s = 611 \exp \left( \frac{17.27T}{237.3 + T} \right) \quad (4)$$

where  $e_s$  is in Pascals and  $T$  is in Celsius.

## 3) MIXING RATIO $w$

Defined as the mass of water vapor  $m_v$  per unit mass of dry air.

$$w = \frac{m_v}{m_d} = \frac{e/(R_v T)}{(p - e)/(R_d T)} = \epsilon \frac{e}{p - e} \quad (5)$$

where  $p = p_d + e$ , with  $p$  the total pressure and  $p_d$  the pressure due to the dry air component. It is often expressed as g/kg. Very moist air might have a mixing ratio of 18 g/kg.

#### 4) SATURATION MIXING RATIO $w_s$

Mixing ratio if the air were saturated. Defined as:

$$w_s = \epsilon \frac{e_s}{p - e_s} \quad (6)$$

#### 5) SPECIFIC HUMIDITY $q$

The specific humidity is defined as the mass of water vapor per unit mass of moist air,

$$q = \frac{m_v}{m_d + m_v} = \frac{e/(R_v T)}{(p - e)/(R_d T) + e/R_v T} = \epsilon \frac{e}{p - (1 - \epsilon)e} \quad (7)$$

#### 6) SATURATION SPECIFIC HUMIDITY $q_s$

$$q_s = \epsilon \frac{e_s}{p - (1 - \epsilon)e_s} \quad (8)$$

Notice that  $q_s$  and  $w_s$  depend on pressure and temperature only - not on the water vapor content of the air.

#### 7) RELATIVE HUMIDITY $RH$

Ratio (expressed as percentage) of the actual mixing ratio to the saturation mixing ratio:

$$RH = 100 \frac{w}{w_s} \approx 100 \frac{e}{e_s} \approx \frac{q}{q_s} \quad (9)$$

Air can have a RH in excess of 100% especially if the air is devoid of aerosols. Such air is supersaturated.

#### 8) VIRTUAL TEMPERATURE

When there is a mixture of water vapor and dry air, it is convenient to retain the gas constant for dry air and use a fictitious temperature called virtual temperature in the ideal gas equation. The density of a mixture of dry air and water vapor is:

$$\rho = \frac{m_d + m_v}{V} = \rho_d + \rho_v = \frac{p - e}{R_d T} + \frac{e}{R_v T} = \frac{p}{R_d T} \left[ 1 - \frac{e}{p} (1 - \epsilon) \right] \quad (10)$$

$$p = \rho R_d T_v \quad (11)$$

where

$$T_v \equiv \frac{T}{1 - \frac{e}{p} (1 - \epsilon)} \quad (12)$$

Virtual temperature is the temperature that dry air would have to attain in order to have the same density as the moist air at the same pressure. Because moist air is less dense than dry air at the same temperature and pressure, the virtual temperature is always greater than the actual temperature by a few degrees. You can obtain an approximate value for the virtual temperature as:

$$T_v \approx (1 + 0.61w)T \approx (1 + 0.61q)T \quad (13)$$

$$R_m \approx (1 + 0.61q)R_d \quad (14)$$

where  $R_m$  is the gas constant for the mixture of dry air and water vapor. We can also obtain the virtual temperature as:

$$\theta_v = T_v \left( \frac{1000}{p} \right)^\kappa \quad (15)$$

## 9) ASCENT AND DESCENT OF MOIST AIR

The specific heats of moist air are different from those of dry air:

$$c_{pm} \approx (1 + .87q)c_{pd} \quad (16)$$

$$c_{vm} \approx (1 + .97q)c_{pd} \quad (17)$$

$$\kappa_m \approx (1 - .26q)\kappa_d \quad (18)$$

$$\theta_m = T \left( \frac{1000}{p} \right)^{\kappa_m} \quad (19)$$

$$(20)$$

Despite the fact that the specific heats of moist air are different than those of dry air (and hence  $\kappa$  is different and the potential temperature  $\theta_m$  is different).  $\theta_m$  differs from  $\theta$  by less than  $0.1^\circ\text{C}$ , *we can treat the unsaturated ascent and descent of moist air as if it were the ascent and descent of dry air and use  $\theta$  instead of  $\theta_m$ . We cannot do this, however, if the moist air is saturated.*

### a. Processes Leading to Saturation

A sample of moist air may undergo a variety of processes that lead to saturation and certain new temperatures are defined that reflect the moisture content of the air.

#### 1) DEW-POINT TEMPERATURE

Temperature to which moist air must be cooled isobarically with  $p$  and  $q$  held constant to just reach saturation with respect to a plane surface of pure water. If air is cooled to  $T_d$ , liquid water will condense on surfaces such as grass and car windows. The dew point temperature is given by:

$$w = w_s(T_d, p) = \epsilon \frac{e_s(T_d)}{p - e_s(T_d)} \quad (21)$$

It follows that the relative humidity at pressure  $p$  and temperature  $T$  is given by:

$$RH = 100 \frac{w_s(T_d, p)}{w_s(T, p)} \quad (22)$$

Equation 21 is a transcendental equation that must be solved graphically or through an iterative process. To do this we can use a calculator that does iterations, a computer or manually using a

Newton-Raphson procedure. The Newton-Raphson procedure is as follows:

- i. Re-write Equation 21 as  $f(T_d) = 0$
- ii. Using Taylor series, we know that  $f(T_d) = f(T_d^0) + f'(T_d^0)[T_d^1 - T_d^0]$  where  $f' = df/dT_d$
- iii. Solve for next estimate  $T_d^1 = T_d^0 - \frac{f(T_d^0)}{f'(T_d^0)}$
- iv. Repeat iteratively to converge to the true solution  $T_d^{n+1} = T_d^n - \frac{f(T_d^n)}{f'(T_d^n)}$

Graphically, the simplest way to find  $T_d$  is using the Skew T-log p chart. The dew-point temperature is always less than or equal to the actual temperature  $T_d \leq T$ .

## 2) FROST-POINT TEMPERATURE $T_f$

Temperature to which moist air must be cooled isobarically, with  $p$  and  $q$  held constant, for the air to reach saturation with respect to a plane surface of pure ice. Given by:

$$w = w_s(T_f, p) = \epsilon \frac{e_{s,i}(T_d)}{p - e_{s,i}(T_d)} \quad (23)$$

This is meaningful only when  $(T_d < 0^\circ\text{C})$ . We will see later that  $e_{s,i}(T) < e_s(T)$

## 3) WET-BULB TEMPERATURE, $T_w$

Temperature to which air may be cooled by evaporating water into it a constant pressure until saturation is reached.  $w$  is not held constant in this process. The first law of Thermodynamics for an isobaric process is:

$$dq = (1 + 0.87w)c_{pd}dT \quad (24)$$

if we neglect the term multiplying  $w$  as it is sufficiently small, we can work with the dry air constants.

$$dq = c_{pd}dT \quad (25)$$

If moist air consists of one gram of dry air and  $w$  grams of moist air, the heat lost by the moist air required to evaporate  $dw$  grams of water is:

$$(1 + w)dq = -Ldw \quad (26)$$

where  $L$  is the latent heat of vaporization. So

$$c_{pd}dT = -\frac{Ldw}{1 + w} \approx -Ldw \quad (27)$$

Neglecting the weak dependence of  $L$  on  $T$  we can integrate 27 from  $(T, r)$  to  $[T_w, w_s(T_w, p)]$  to obtain:

$$T_w = T - \frac{L}{c_{pd}}(w_s(T_w, p) - r) \quad (28)$$

This is also a transcendental equation that must be solved iteratively. The wet bulb temperature cannot be found on a skew T-log p diagram.

#### 4) EQUIVALENT TEMPERATURE $T_e$

Temperature a sample of moist air would attain if all the moisture were condensed out at constant pressure. We obtain the expression for  $T_e$  by integrating equation 27 from  $(T, w)$  to  $(T_e, 0)$ .

$$T_e = T + \frac{L}{c_{pd}}w \quad (29)$$

Thus,  $T_e \geq T$

#### 5) ADIABATIC SATURATION TEMPERATURE $T_s$

Temperature at which saturation is reached when moist air is cooled adiabatically with  $w$  held constant. Easily understood in a skew T-log p diagram. If an parcel ascends along its adiabat which has potential temperature  $\theta$ , both temperature and pressure decrease. The saturation vapor pressure decreases more rapidly than  $p$ , so the mixing ratio decreases. Eventually, the air parcel will reach a point where the decreasing saturation mixing ratio just equals the mixing ratio of the air parcel.

$$w = w_s(T_s, p_s) = \epsilon \frac{e_s(T_s)}{p_s - e_s(T_s)} \quad (30)$$

where  $p_s$  is the adiabatic saturation pressure and  $T_s = T(p_s/p)^\kappa$ , is the adiabatic saturation temperature. This is, yet another, transcendental equation which must be solved iteratively. We may speak interchangeably of condensation and saturation at a point. The level at which an air mass attains saturation by adiabatic ascent is called the **lifting condensation level LCL**. We can obtain the pressure of the LCL as

$$p_s = p \left( \frac{T_s}{T} \right)^{1/\kappa} \quad (31)$$

We can find the altitude of the LCL using the hypsometric equation between  $p$  and  $p_s$