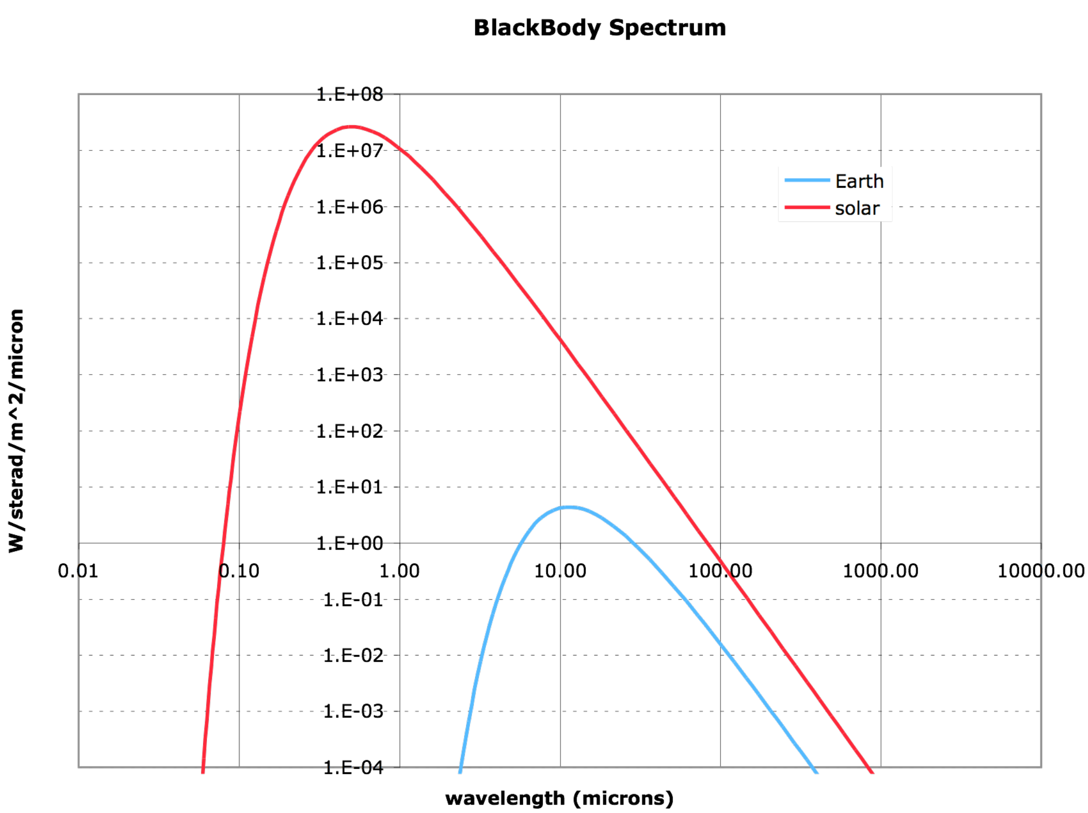
Nomenclature and Definition of Radiation Quantities

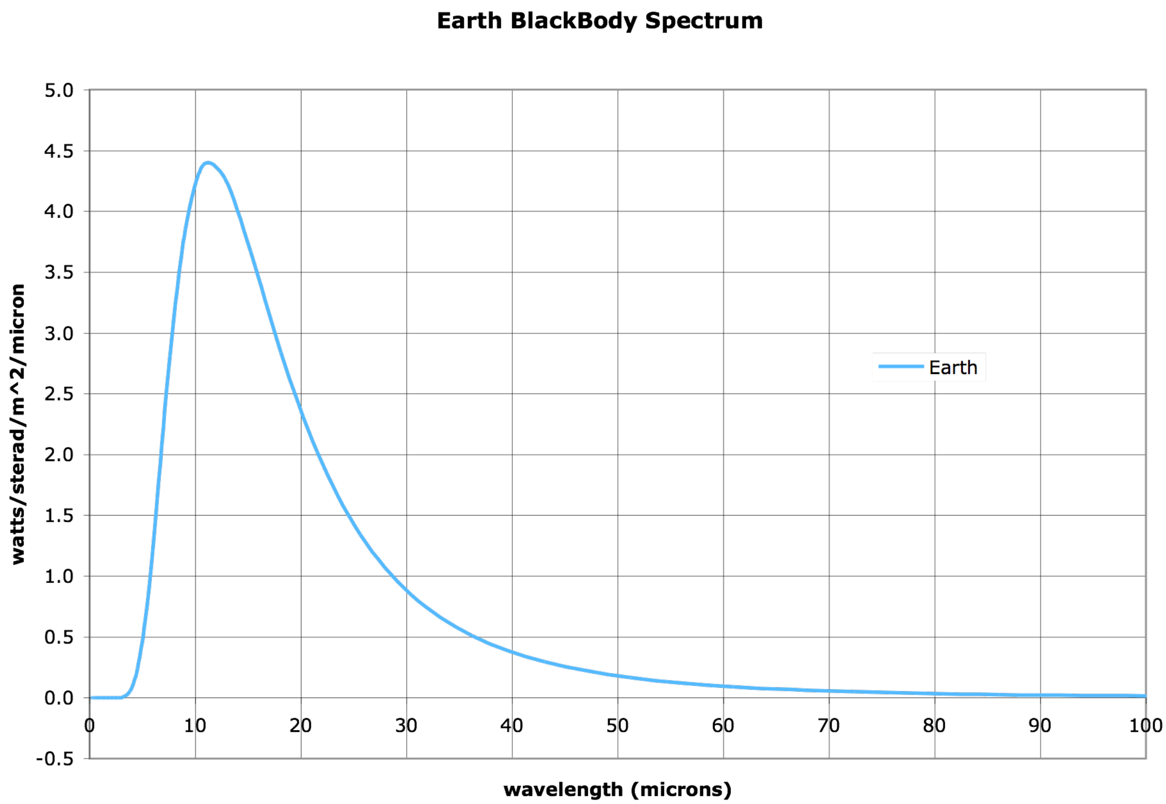
The various Radiation Quantities are defined in Table 2-1. Keeping them straight is difficult and the meanings may vary from textbook to textbook. I recommend using the table and in particular the units to keep the various terms straight. Angel Otarola pointed to a web page relevant to this topic: See "Getting Intense about Intensity" http://www.optics.arizona.edu/Palmer/intenopn.html

I also below discuss an ambiguity in the blackbody equation depending on whether you are talking about radiant flux density or radiance.

Blackbody radiation

A perfect blackbody absorbs all radiation that strikes it. It emits "thermal" radiation described by the Planck Function. It is so named because Max Planck was the first to produce a theoretical explanation of this radiation spectrum in 1901. It had been noted that Blackbody radiation can be produced by a small hole or opening in a cavity whose insides absorb all radiation striking the cavity. The radiation spectrum strongly depended on the temperature of the cavity. In order to find a function that fit the observed blackbody emission spectrum, Planck had to assume that the oscillations of the radiation in the cavity were quantized. This realization was a major step in creating of the field of physics we know as quantum mechanics.





The different forms of the Planck function

There are several different forms of the Planck function. First of all, it can be written in terms of frequency, ν, as $B(\nu,T)$ or wavelength, λ, $B(\lambda,T)$. Conversion between the frequency and wavelength forms is accomplished via the equation,

$$B_\nu(\nu,T)\,\mathrm{d}\nu = B_\lambda(\lambda,T)\,\mathrm{d}\lambda.$$

Furthermore, it can be written in a radiance form (watts/steradian/m^2 per Hz or per wavelength), a radiant flux density (also known as spectral emittance) form (watts/m^2 per Hz or per wavelength) or a radiant energy density form (joules/m^3 per Hz or per wavelength). There are other versions as well and you must check the units to understand the form.

The **radiance form** *as a function of frequency* with watts/steradian/m^2/Hz is

$$B(\nu,T) = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT}-1}$$

Check units: J s $s^2/(s^3 m^2)$ = J/m^2 = J/m^2/s/Hz = W/m^2/Hz. The problem is you can't see the steradians .

The Planck function written in the **radiant energy flux density form** *as a function of frequency* form (watts/m^2/Hz) is

$$B_{flux}(\nu,T) = \frac{2\pi h\nu^3}{c^2}\frac{1}{e^{h\nu/kT}-1}$$

The factor of π comes from integrating out the angular dependence (see below).

Another form is the frequency dependent form of the **radiant energy density form** in joules per m^3 per Hz. We take the radiance form, multiply it by 4π to integrate out the steradian dependence and divide it by c because flux density equals density times the speed of propagation.

$$B(\nu,T) = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/kT}-1}$$

Check units: J s $s^3/(s^3\ m^3)$ = $J/m^3/Hz$.

EvZ write the Planck function in terms of wavelength in the **radiant flux density** or **spectral emittance** form (watts/m^2/wavelength).

$$S_{flux}(\lambda,T) = \frac{2\pi hc^2}{\lambda^5}\frac{1}{e^{ch/\lambda kT}-1}$$

Again, this form has an extra factor of π in comparison to the radiance form (watts/steradian/m^2/Hz) for reasons we'll show below.

Relation between Radiant Flux Density (watts/m²) and Radiance (watts/m²/steradian)

Suppose we want to know the upward energy flux density from a surface. Each surface element is radiating in all directions, each containing an upward component. We want to add up all of the vertical components from energy radiated in all directions. So we integrate over the upward-looking hemisphere.

$$M(\nu,T) = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/2} B(\nu,T)\ \cos\phi\frac{r\sin\phi d\theta\ rd\phi}{r^2} = B(\nu,T)\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/2}\cos\phi\sin\phi d\phi\ d\theta$$

$$M(\nu,T) = B(\nu,T)\left.\frac{\sin^2\phi}{2}\right|_0^{\pi/2}\left.\theta\right|_0^{2\pi} = B(\nu,T)\left(\frac{1}{2}-0\right)(2\pi-0) = \pi\ B(\nu,T)$$

So the factor of π that you will see in radiant flux density equations comes from doing a hemispherical integration of the upward (or downward) component of radiance vector (which has a per steradian dependence) when the radiance vector is isotropic, that is it has no directional dependence.

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Wien's (displacement) law

The peak of the emission from a blackbody (in the wavelength form) and its temperature are related as

$$\lambda_{\max}=\frac{b}{T}$$

where λ_{max} is the wavelength (in m) at which spectral emission is maximum and b is about 2.90×10^{-3} m K.

The peak of the frequency form of the Planck function occurs at $2.82 = h\nu/kT$ which is about a factor of 2 longer wavelength larger than the peak when the equation is written in the wavelength form.

Stephen Boltzmann law

The Stephan-Boltzmann law gives the total flux density from the Planck function integrated across the spectrum. The result is

$$F = \sigma T^4$$

where $\sigma = 5.67\text{x}10^{-8}$ watts/m^2/Kelvin^4. One can remember the constant as 5678 with the last digit being the exponent and the form of mantissa being #.##.

To go from Planck function to the Stephan Boltzmann law, you integrate the Planck function across the spectrum. See: http://scienceworld.wolfram.com/physics/Stefan-BoltzmannLaw.html.

The Stephan Boltzmann law is remarkable because the form is so simple, the results depends only on temperature and the dependence on temperature is so strong. The surface of the Earth is 20 times colder at 300K than the surface of the sun (6000K) but the sun puts out $20^4 = 1.6\text{x}10^5$ times as much radiant energy per square meter as does the Earth. Furthermore the radius of the Sun is ~10x the radius of Jupiter which is ~10x the radius of the Earth. So the total black body radiant energy emitted by the Sun is about $10^4\ \text{x}1.6\text{x}10^5 = 1.6\text{x}10^9$ times that of the Earth.

Reflection, absorption, transmission and emission

EM radiation striking an object is either reflected, absorbed or transmitted through the object. The albedo, *A*, is the fraction of incident light that is reflected. The visible albedo of the Earth is about 0.3 due largely to clouds.

If the object is very thick so there is no EM transmission through the object, then the fraction of incident light absorbed, which is called absorptivity, is 1-*A*.

Kirchoff's law:

Based on energy balance, the emissivity of an object, that is how close the object's emission is to a perfect blackbody, must equal the absorptivity. Otherwise objects could reach infinitely high or infinity low temperatures. So a good absorber is a good emitter. A good reflector is a poor emitter.

These properties are very wavelength dependent. Snow is an excellent reflector and therefore poor absorber and emitter at visible wavelengths. However, at IR wavelengths, snow is an excellent absorber and emitter and therefore poor IR reflector.

Einstein coefficients, the Boltzmann distribution and the Planck function

The Boltzmann distribution states that the number of molecules, N_i, in a state of energy, E_i, for a system with temperature, T, that is in thermodynamic equilibrium is proportional to

$$N_i \sim e^{-Ei/kT}$$

Degeneracy It is not equal to because the number depends on the total number of available states and molecules. Assume we have an electromagnetic wave propagating through a material in which two of the energy levels, i and j are such that

$$h\nu = E_j - E_i$$

The wave will then excite some of the population of level *i* up to level *j* and lose some if its energy is doing so, having transferred this energy to the material. The rate at which molecules are moved spontaneously from the upper state to the lower state is

$$\left.\frac{dN_i}{dt}\right|_{spont} = A_{ji}N_j$$

and the rate the system loses energy spontaneously is therefore

$$\left.\frac{dE_{sys}}{dt}\right|_{spont} = -\left.\frac{dN_i}{dt}\right|_{spont} h\nu = -A_{ji}N_j h\nu$$

The molecules absorb some of the photons in the incident beam which knocks molecules from the *i*th energy state up to the *j*th state. That rate is

$$\left.\frac{dN_i}{dt}\right|_{abs} = -B_{ij}N_i I(\nu)$$

The rate of energy lost from the beam is simply

$$\frac{dI(v)}{dt} = -hv\frac{dN_i}{dt}\bigg|_{abs} = -hvB_{ij}N_iI(v)$$

What Einstein realized in 1917 I believe was that in order to have radiative-energy equilibrium that satisfies the Boltzmann distribution and the Planck function there has to be a third radiation interaction with matter term, where the incident EM radiation causes molecules to move from the higher state to the lower energy state:

$$\frac{dN_i}{dt}\bigg|_{stim} = +B_{ji}N_jI(v)$$

With this added *stimulated* emission term, equilibrium can be achieved. The stimulated emission term adds energy to the incident EM beam:

$$\frac{dI(v)}{dt} = +hv\frac{dN_i}{dt}\bigg|_{stim} = +hvB_{ji}N_jI(v)$$

In equilibrium the number of molecules in state I must remain unchanged. So

$$\frac{dN_i}{dt}\bigg|_{spon} + \frac{dN_i}{dt}\bigg|_{abs} + \frac{dN_i}{dt}\bigg|_{stim} = 0$$

$$A_{ji}N_j - B_{ij}N_iI(v) + B_{ji}N_jI(v) = 0$$

The Boltzmann distribution constraint is

$$\frac{N_i}{N_{tot}} = \frac{g_ie^{-E_i/kT}}{Z}$$

where Ntot is the total number of molecules, g_i is the number of states with energy, E_i, which is called the degeneracy, and Z is the partition function which is the sum of the g_i $e^{-Ei/kT}$ terms for all possible states which is needed to normalize the N_i/N_{tot} ratio. The Boltzmann distribution gives us the ratio, N_j/N_i

$$\frac{N_j}{N_i} = \frac{g_je^{-E_j/kT}}{g_ie^{-E_i/kT}}$$

Plugging this in yields

$$A_{ji}N_i\frac{g_j}{g_i}e^{-(E_j-E_i)/kT} - B_{ij}N_iI(v_{ij}) + B_{ji}N_i\frac{g_j}{g_i}e^{-(E_j-E_i)/kT}I(v_{ij}) = 0$$

$$A_{ji}N_i\frac{g_j}{g_i}e^{-hv_{ij}/kT} - B_{ij}N_iI(v_{ij}) + B_{ji}N_i\frac{g_j}{g_i}e^{-hv_{ij}/kT}I(v_{ij}) = 0$$

Now the final piece is we need to plug in the Planck function for $I(v)$. The question is which form do we use. The answer is we can use either the flux or energy density forms with the understanding that it alters the dimensions of B_{ij} and B_{ji}. To generalize we write the Planck function as

$$B(\nu,T) = F(\nu)\frac{1}{e^{h\nu/kT}-1}$$

where F(ν) = $\frac{2h\nu^3}{c^2}$ using the radiance form and F(ν) = $\frac{8\pi h\nu^3}{c^3}$ using the radiant energy density form.
Note that this *B(ν,T)* is different from the B_{ij}'s that are the Einstein coefficients

$$A_{ji}N_i\frac{g_j}{g_i}e^{-h\nu_{ij}/kT} - B_{ij}N_iF(\nu_{ij})\frac{1}{e^{h\nu_{ij}/kT}-1} + B_{ji}N_i\frac{g_j}{g_i}e^{-h\nu_{ij}/kT}F(\nu_{ij})\frac{1}{e^{h\nu_{ij}/kT}-1} = 0$$

$$A_{ji}\frac{g_j}{g_i}\left[1-e^{-h\nu_{ij}/kT}\right] - B_{ij}F(\nu_{ij}) + B_{ji}\frac{g_j}{g_i}e^{-h\nu_{ij}/kT}F(\nu_{ij}) = 0$$

This relation must hold for all temperatures. The only way this is possible is for $B_{ij}\, g_i = B_{ji}\, g_j$ such that

$$A_{ji}\frac{g_j}{g_i}\left[1-e^{-h\nu_{ij}/kT}\right] = +B_{ij}F(\nu_{ij})\left[1-e^{-h\nu_{ij}/kT}\right]$$

from which we also see the relation between A_{ji} and B_{ij},

$$\frac{A_{ji}}{B_{ij}} = \frac{g_i}{g_j}F(\nu_{ij})$$

So in the ***radiance*** form,

$$A_{ji} = \frac{g_i}{g_j}\frac{2h\nu_{ij}^3}{c^2}B_{ij}$$

In the radiant energy density form,

$$A_{ji} = \frac{g_i}{g_j}\frac{8\pi h\nu_{ij}^3}{c^3}B_{ij}$$

This is the form used by EvZ except they include the index of refraction to account for the fact that the speed of light in the medium is not the same as the speed of light in a vacuum, v = c/n.

$$A_{ji} = \frac{8\pi h\nu^3 n^3}{c^3}B_{ji} = \frac{8\pi h n^3}{\lambda^3}B_{ji}$$

Now we can understand what is happening to our incident EM beam as it propagates through the medium.

EvZ write the rate, p_{ij}, at which energy is lost from the wave is

$$p_{ij} = B_{ij}\,\varepsilon_\nu$$

where ε_ν is the wave *energy density* per unit frequency and B_{ij} is a constant determined by the system. Note that EvZ consider the problem in terms of radiant energy density which is why they use the radiant energy density form of the Planck function.

The rate of spontaneous emission from level j to level i is described as

$$p_{ji} = A_{ji}$$

The rate of stimulated emission where the incident radiation causes some of the population to transition from level j to level i is

$$p_{ji} = B_{ji}\ \varepsilon_\nu$$

Now, consider the rate at which signal energy is lost into and gained from the medium. Note that stimulated emission adds energy to the signal. The change in the energy density of the signal is

$$(N_j B_{ji} - N_i B_{ij})\ \varepsilon_\nu = -(N_i - N_j)\ B_{ji}\ \varepsilon_\nu$$

Now, in thermodynamic equilibrium, the population is in a Boltzmann distribution, we can write this as

$$-N_i\ (1 - e^{-(Ej-Ei)/kT})\ B_{ji}\ \varepsilon_\nu = -(1 - e^{-h\nu/kT})\ N_i\ B_{ji}\ \varepsilon_\nu$$

This normally results in energy lost from the electromagnetic beam because in thermodynamic equilibrium, $N_i > N_j$ and we can write N_i - N_j as $N_i\ (1 - e^{-h\nu/kT}) > 0$ and the equation above is negative so there is an energy loss from the beam.

We have gone through this explanation because the strength of the absorption lines is directly related to these Einstein coefficients. We will make that connection a bit later. The possibility of inverted populations and using the medium to amplify an incident signal has led to lasers.

LASERs & MASERs: Light Amplification by Stimulated Emission Radiation (LASERs) and Microwave Amplification by Stimulated Emission Radiation (MASERs) work by inverting the population through some pumping mechanism so that the population in state N_j is larger than that in state N_i. Under these conditions, the stimulated emission will be larger than the absorption and the incident signal will be amplified as it passes through the medium.

$$(N_j B_{ji} - N_i B_{ij})\ \varepsilon_\nu = (N_j - N_i)\ B_{ji}\ \varepsilon_\nu > 0 \quad \text{so} \quad \varepsilon_{\nu out} > \varepsilon_{\nu in}$$