Limb viewing weighting functions
Straight line approx

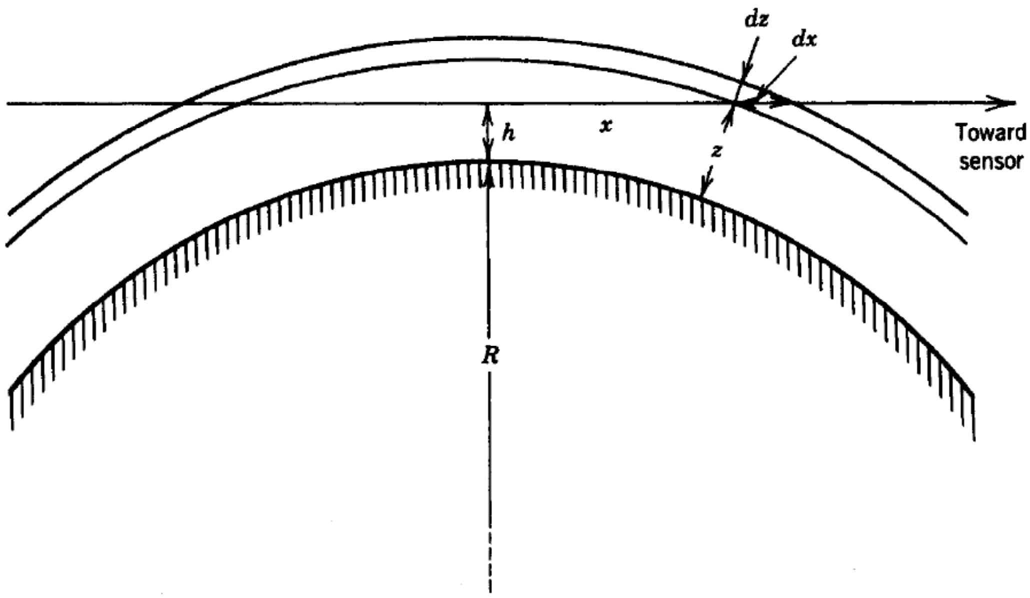

Figure 9-11. Geometry for a limb sounder.

assuming straight line signal paths,
$(R+z)^2 = (R+h)^2 + x^2$ => $2(R+z)\,dz = 2x\,dx$ => $(R+z)\,dz = x\,dx$

$$dx = \frac{(R+z)}{x}dz = \frac{(R+z)}{\sqrt{(R+z)^2 - (R+h)^2}}dz = \left(1 - \frac{(R+h)^2}{(R+z)^2}\right)^{-1/2} dz = g(z,h)dz$$

$\Delta B = B(v,T)\,\alpha(v,z)\,dx = B(v,T)\,\alpha(v,z)\,g(z,h)\,dz$

Assuming local spherical symmetry, the effective spectral emittance from a layer observed by the sensor is

$$\Delta B_e = \Delta B\left(e^{-\tau_1(v,z)} + e^{-\tau_2(v,z)}\right)$$

where the near side optical depth is

$$\tau_1(v,z) = \int_z^\infty \alpha(v,\zeta)g(\zeta,h)d\zeta$$

and the far side optical depth is

$$\tau_2(v,z) = \int_h^z \alpha(v,\zeta)g(\zeta,h)d\zeta + \int_h^\infty \alpha(v,\zeta)g(\zeta,h)d\zeta$$

Putting the various terms together we get

$$B_a(v) = \int_h^\infty \alpha(v,z)B[v,T(z)]g(z,h)\left(e^{-\tau_1(v,z)} + e^{-\tau_2(v,z)}\right)dz$$

$$T_a(v,h) = \int_h^\infty \alpha(v,z)T(z)g(z,h)\left(e^{-\tau_1(v,z)} + e^{-\tau_2(v,z)}\right)dz$$

The left term is the near side contribution and the right term is the far side contribution. Notice there is no surface contribution because the surface does not fall within the beam. As with the nadir sounder, the atmospheric temperature emission can be written in terms of a weighting function.

$$T_a(v,h) = \int_0^\infty W(v,h,z)T(z)dz$$

where the weighting function is given by

$$W(v,h,z) = \alpha(v,z)g(z,h)\left(e^{-\tau_1(v,z)} + e^{-\tau_2(v,z)}\right) \quad \text{for } z \geq h$$

$$W(v,h,z) = 0 \quad \text{for } z < h$$

Let's look at these weighting functions and get first order understanding of their behavior. To do so, we assume the simple exponential form for α that we have used previously.

$$T_a(v,h) = \int_h^\infty \alpha(v,z)T(z)g(z,h)\left(e^{-\tau_1(v,z)} + e^{-\tau_2(v,z)}\right)dz$$

where the near side optical depth is

$$\tau_1(v,z) = \int_z^\infty \alpha(v,\zeta)g(\zeta,h)d\zeta$$

and the far side optical depth is

$$\tau_2(v,z) = \int_h^z \alpha(v,\zeta)g(\zeta,h)d\zeta + \int_h^\infty \alpha(v,\zeta)g(\zeta,h)d\zeta$$

Plug in the exponential form $\alpha(z) = \alpha_0\exp(-z/H)$ into the near side optical depth

$$\tau_1(v,z) = \int_z^\infty \alpha_0(v)\exp(-\zeta/H)\frac{(R+\zeta)}{\sqrt{(R+\zeta)^2 - (R+h)^2}}d\zeta$$

Use the approximation that $z = x^2/2R$.

$$\tau_1(v, z = x^2/2R) = \int_{\sqrt{2zR}}^\infty \alpha_0(v)\exp(-\psi^2/2RH)d\psi$$

This is closely related to the *erfc* function:

$$erfc(x) = 1 - erf(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty \exp(-t^2)dt$$

with $y^2 = x^2/2RH$. So $y = x/(2RH)^{½}$ and $dx = (2RH)^{½}\, dy$ and, when $x = (2zR)^{1/2}$, $y = (2zR)^{1/2}/(2RH)^{½} = (z/H)^{1/2}$. So we can write τ_1 in terms of an *erfc* function as follows

$$\tau_1(v, z = x^2/2R) = \alpha_0(v)\sqrt{2RH}\int_{\sqrt{z/H}}^{\infty}\exp(-y^2)dy = \alpha_0(v)\sqrt{\frac{\pi RH}{2}}\, erfc\left(\sqrt{z/H}\right)$$

$$\tau_1(v,z) = \alpha_0(v)\sqrt{\frac{\pi RH}{2}}\, erfc\left(\sqrt{z/H}\right) = \alpha_0(v)\sqrt{\frac{\pi RH}{2}}\left[1 - erf\left(\sqrt{z/H}\right)\right]$$

Now let's do the same for the far side optical depth, τ_2.

$$\tau_2(v,z) = \int_h^z \alpha(v,\zeta)g(\zeta,h)d\zeta + \int_h^{\infty}\alpha(v,\zeta)g(\zeta,h)d\zeta$$

Plug in the exponential form $\alpha(z) = \alpha_0 \exp(-z/H)$ into the far side contribution to optical depth

$$\tau_2(v,z) = \int_0^{\sqrt{2zR}} \alpha_0(v)\exp\left(-\frac{\psi^2}{2RH}\right)d\psi + \int_0^{\infty}\alpha_0(v)\exp\left(-\frac{\psi^2}{2RH}\right)dx$$

$$\tau_2(v,z) = \alpha_0(v)\sqrt{2RH}\int_0^{\sqrt{z/H}}\exp\left(-y^2\right)dy + \alpha_0(v)\sqrt{2RH}\int_0^{\infty}\exp\left(-y^2\right)dy$$

$$\tau_2(v,z) = \alpha_0(v)\sqrt{\frac{\pi RH}{2}}\left[erf\left(\sqrt{z/H}\right) + 1\right]$$

With these, equations for τ_1 and τ_2, we can plot the weighting function behavior versus altitude. The next plot shows the weighting functions with frequencies optimized for the weighting functions to peak every 4 km.

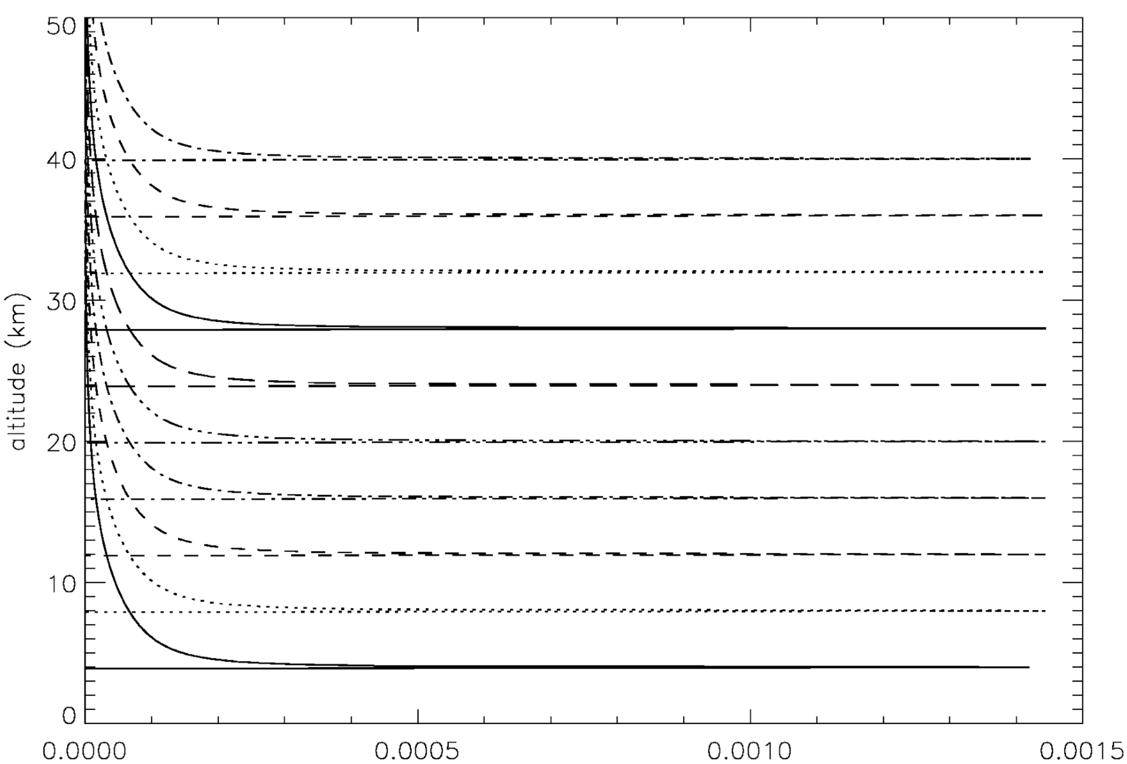

Figure: Microwave limb viewing weighting functions

The next figure shows the weighting functions for a fixed viewing angle and therefore tangent altitude where the atmosphere is sampled by a set of frequencies such that the absorption coefficient at the tangent point increases by a factor of 2 for each frequency closer to the line center.

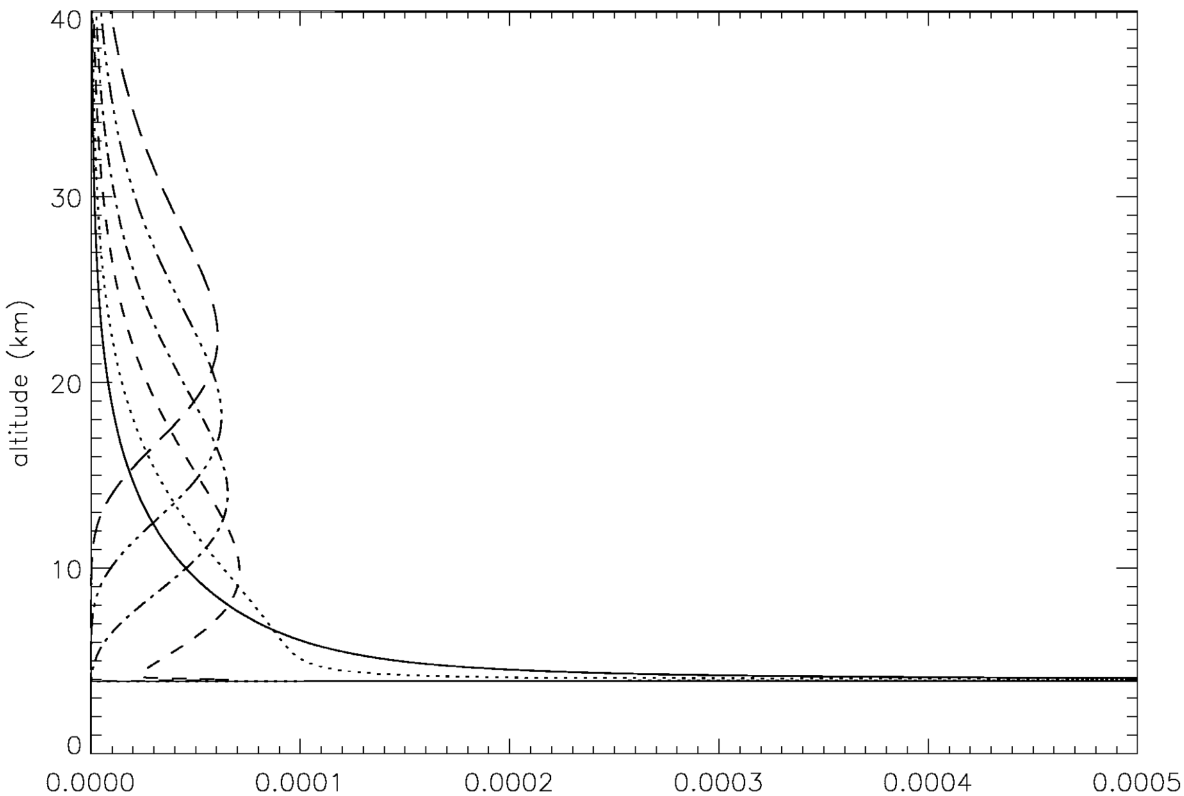

Figure: Weighting functions of a fixed viewing geometry with a range of frequencies.

Effect of Antenna Pattern

Note that to realistically represent what the instruments actually measure, these weighting functions need to be convolved with the antenna pattern of the limb viewing instrument which smoothes out the weighting function a bit.