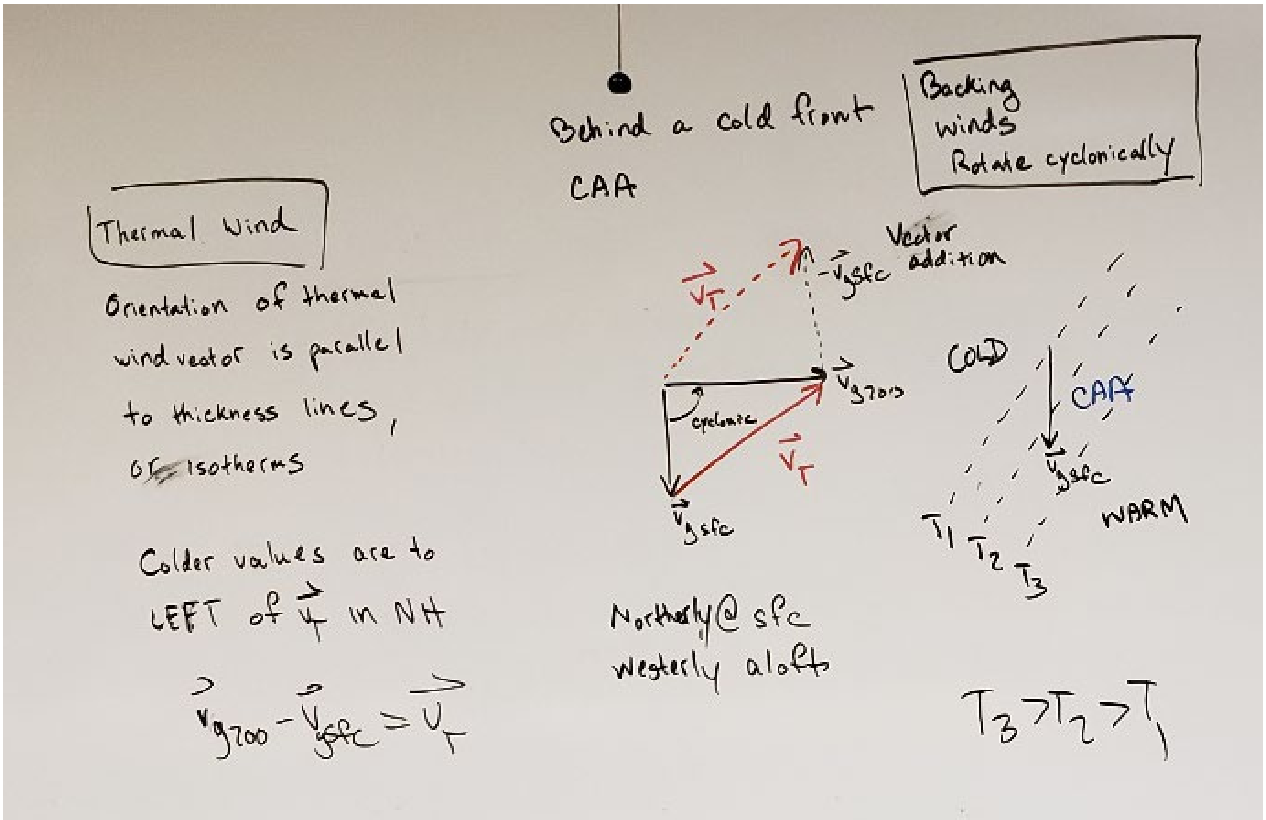
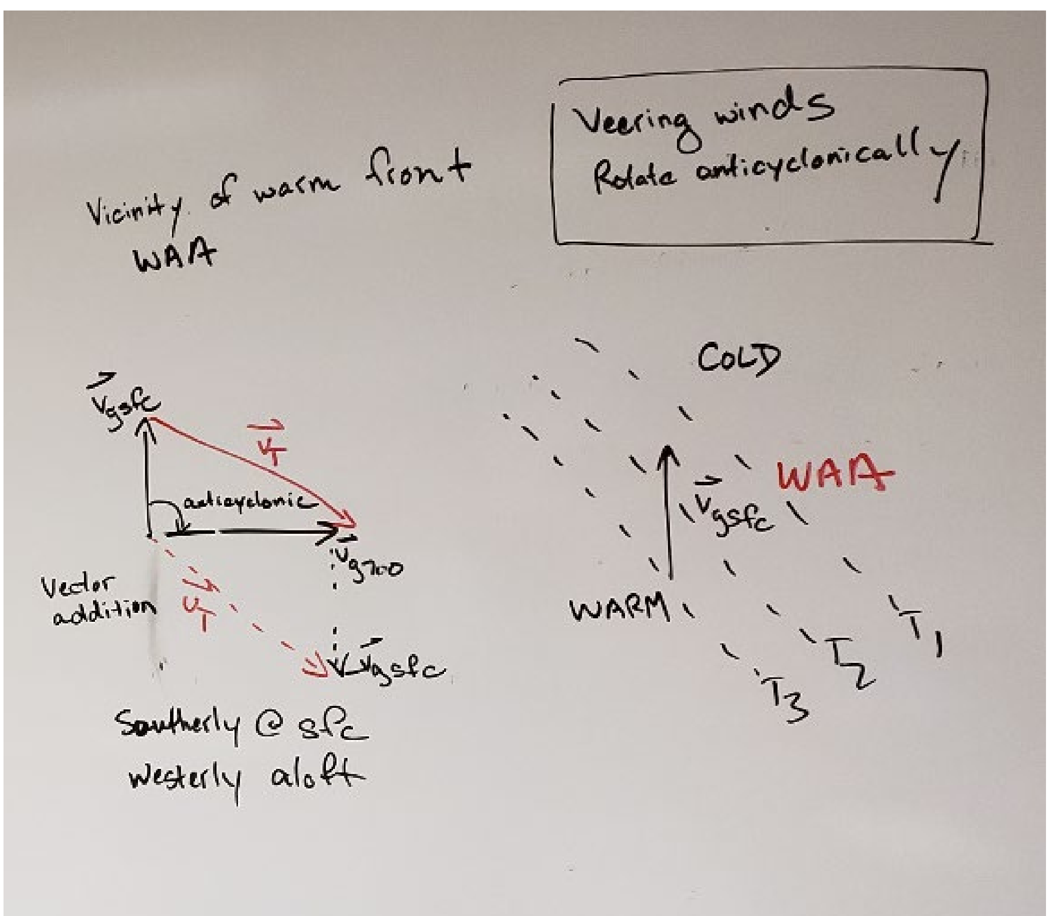


Behind a cold front
CAA
Backing winds
Rotate cyclonically
Thermal wind
Orientation of thermal wind vector is parallel to thickness lines, or isotherms
Colder values are to LEFT of $\vec{V}_T$ in NH
$\vec{v}_{g700} - \vec{V}_{gsfc} = \vec{V}_T$
Vector addition
$\vec{V}_T$
$-\vec{V}_{gsfc}$
$\vec{v}_{g700}$
cyclonic
$\vec{V}_T$
$\vec{v}_{gsfc}$
Northerly @ sfc
westerly aloft
COLD
CAA
$\vec{V}_{gsfc}$
WARM
T_1 T_2 T_3
$T_3 > T_2 > T_1$



Vicinity of warm front
WAA
Veering winds
Rotate anticyclonically
$\vec{V}_{gsfc}$
$\vec{V}_T$
anticyclonic
$\vec{v}_{g700}$
Vector addition
$\vec{V}_T$
$-\vec{V}_{gsfc}$
Southerly @ sfc
westerly aloft
COLD
WAA
$\vec{V}_{gsfc}$
WARM
T_1
T_2
T_3

Vorticity

In layman's terms a measure of spin in the atmosphere. Maximum in vorticity often referred to (inarticulately) as a "piece of energy" by TV meteorologists.

Mathematically, curl of the wind

$$\nabla \times \vec{V} = \vec{\zeta} \qquad \vec{V} = (u, v, w)$$

A vector quantity

$$\nabla \times \vec{V} = \left|\begin{array}{ccc} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & v & w \end{array}\right| \begin{array}{cc} \hat{i} & \hat{j} \\ \frac{\partial}{\partial x} & \partial/\partial y \\ u & v \end{array}$$

$$= \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{j}$$

$$+ \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}$$

Three components!

In synoptic meteorology because we assume $W << u, v$, we are typically concerned with just the vertical ($\hat{k}$) component of vorticity

$$\hat{k} \cdot (\nabla \times \vec{V}) = \zeta_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \rightarrow \text{Relative (local) vorticity}$$

Can estimate on a Wx map (upper air chart) by a finite difference approach.

<u>Vorticity</u> <u>equation</u> (tendency)

$$\underbrace{\frac{\partial \zeta_z}{\partial t}}_{A} = \underbrace{-\vec{V} \cdot \nabla \zeta_z}_{\times B} \underbrace{- \omega \frac{\partial \zeta_z}{\partial p}}_{C} - \underbrace{\left[\frac{\partial \omega}{\partial x} \frac{\partial v}{\partial p} - \frac{\partial \omega}{\partial y} \frac{\partial u}{\partial p} \right]}_{D}$$

$$\underbrace{+ \zeta_z \frac{\partial \omega}{\partial p}}_{\times E} + \underbrace{\left[\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right]}_{F}$$

$\times$ = Relevant for synoptic scale

Other terms <u>ONLY</u> important for mesoscale.

A = local time rate of change term

B = Horizontal vorticity advection
(PVA or NVA)

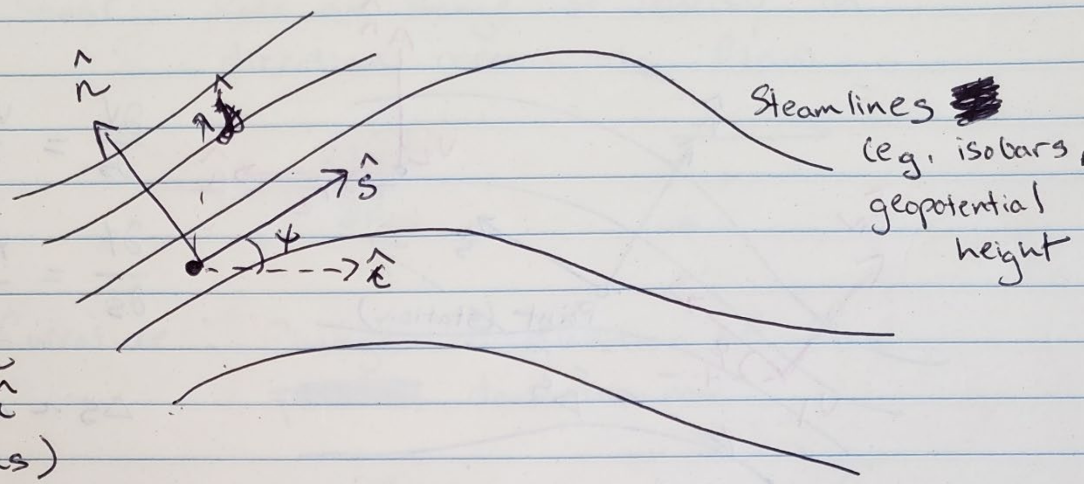
C = Vertical vorticity advection

D = Tilting of horizontal vorticity to
vertical

E = Vortex stretching (due to terrain,
diabatic heating

F = Friction

Natural coordinate system



ψ = angle between $\hat{s}$ and $\hat{i}$ (radians)

$\hat{n}$ = vector normal to streamlines
$\hat{s}$ = vector parallel to streamlines

Vorticity and divergence

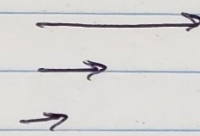
Vorticity $\quad \hat{k}\cdot\nabla\times\vec{V} \quad |\vec{V}|\dfrac{\partial\psi}{\partial s} - \dfrac{\partial|\vec{V}|}{\partial n} \quad \zeta_z = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}$

Divergence $\quad \nabla\cdot\vec{V} \quad |\vec{V}|\dfrac{\partial\psi}{\partial n} + \dfrac{\partial|\vec{V}|}{\partial s} \quad \mathrm{DIV} = \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y}$

Other important kinematic flow properties

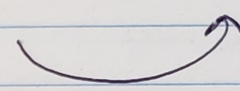
Shear: Rate of change of velocity in direction normal to flow

$$\sim \frac{\partial |\vec{V}|}{\partial n}$$



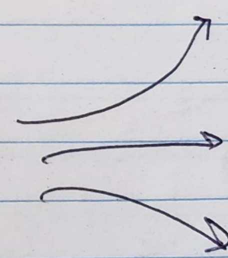
Curvature: Change in direction of flow downstream

$$|\vec{V}| \frac{\partial \psi}{\partial s}$$



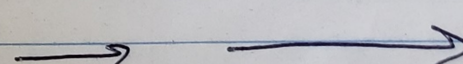
Diffluence: Rate of Change of the direction of flow normal to motion

$$|\vec{V}| \frac{\partial \psi}{\partial n}$$



Stretching: Rate of change of flow in downstream direction

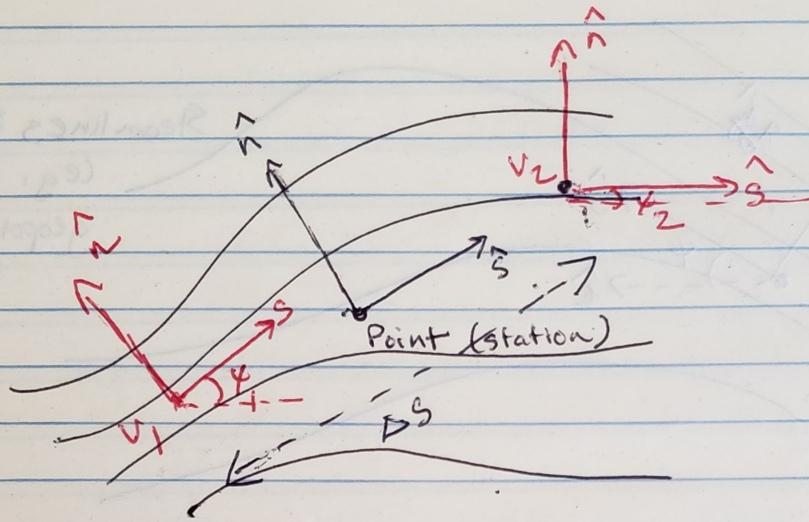
$$\frac{\partial |\vec{V}|}{\partial s}$$



Vorticity = Curvature + Shear

Divergence = Diffluence + Stretching.

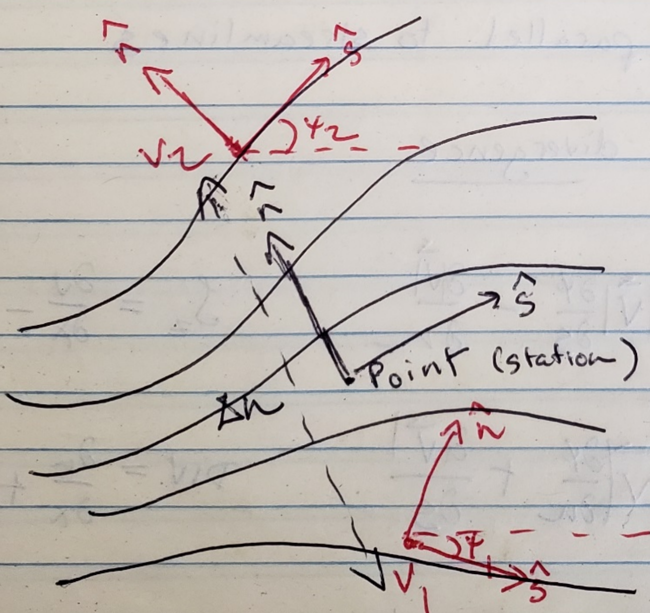
Suggested procedure for calculations of ψ derivative terms in natural coordinates:



$$\frac{\partial V}{\partial s} = \frac{V_2 - V_1}{\Delta s}$$

$$\frac{\partial \psi}{\partial s} = \frac{\psi_2 - \psi_1}{\Delta s}$$

$\Delta s \sim 1000\,\mathrm{km}$



$$\frac{\partial V}{\partial n} = \frac{V_2 - V_1}{\Delta n}$$

$$\frac{\partial \psi}{\partial n} = \frac{\psi_2 - \psi_1}{\Delta n}$$

$\Delta n \sim 1000\,\mathrm{km}$

ψ angle in radians

Quasi-geostrophic theory

Basic idea: What is a way we can account for the effect of advection by the geostrophic wind on vertical motion?

QG will provide us an equation that we can derive vertical motion based on synoptic-scale criteria: These are:

- Vorticity advection (PVA, NVA)
- Temperature advection (WAA, CAA)
- Diabatic heating ($\dot{Q}$)

Again, where QG applies corresponds well with our definition of synoptic-scale we've been developing so far.

- Small Rossby number ($R_0 < 0.1$)
- Ageostrophic wind about 10% of V_g
- Adiabatic, frictionless
- Uniform static stability
- Hydrostatic balance.

Derivation of QG Vorticity Equation

Start with full equation of motion:

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} - \omega\frac{\partial u}{\partial p} - \frac{\partial \Phi}{\partial x} + fv$$

$$\frac{\partial v}{\partial t} = -u\frac{\partial v}{\partial x} - v\frac{\partial v}{\partial y} - \omega\frac{\partial v}{\partial p} - \frac{\partial \Phi}{\partial y} - fu$$

Quasi-geostrophic system:

$$\frac{d}{dt_g} = \frac{\partial}{\partial t} + \vec{V}_g \cdot \nabla_h = \frac{\partial}{\partial t} + u_g\frac{\partial}{\partial x} + v_g\frac{\partial}{\partial y}$$

Consider variation of f with latitude

$$f = f_0 + {}^{\partial f}/_{\partial y}\, y \quad \underline{\text{or}} \quad f_0 + \beta y$$

$$\underbrace{\frac{d\vec{V}_g}{dt_g}}_{\text{Total derivative}} = -\nabla\Phi - (f_0 + \beta y)\,\hat{k} \times \underbrace{\vec{V}}_{\substack{(\vec{V}_g + \vec{V}_{ag}) \\ \text{geostrophic} \\ + \\ \text{ageostrophic}}}$$

Using definition of geostrophic wind

$$\vec{v}_g = \frac{1}{f_0}\hat{k} \times \nabla\Phi$$

$$f_0\hat{k} \times \vec{v}_g = -\nabla\Phi$$

Substitute for $-\nabla\Phi$ in equation for $d\vec{v}_g/dt_g$

$$\frac{d\vec{v}_g}{dt_g} = f_0(\hat{k} \times \vec{v}_g) - (f_0 + \beta y)\hat{k} \times (\vec{v}_g + \vec{v}_{ag})$$

Terms with $f_0(\hat{k} \times \vec{v}_g)$ cancel

$$\frac{d\vec{v}_g}{dt_g} = -f_0 \times \vec{v}_{ag} - \overset{0}{\cancel{\beta y \hat{k} \times \vec{v}_{ag}}} - \beta y \hat{k} \times \vec{v}_g$$

To zero
by scaling

$$\frac{d\vec{v}_g}{dt_g} = -f_0\hat{k} \times \vec{v}_{ag} - \beta y \hat{k} \times \vec{v}_g$$

Expanding into components

(ueq) $$\frac{\partial u_g}{\partial t} + u_g\frac{\partial u_g}{\partial x} + v_g\frac{\partial u_g}{\partial y} - f_0 v_{ag} - \beta y v_g = 0$$

(veq) $$\frac{\partial v_g}{\partial t} + u_g\frac{\partial v_g}{\partial x} + v_g\frac{\partial v_g}{\partial y} + f_0 u_{ag} + \beta y u_g = 0$$

Construct QG vorticity (ζ_g) by:

$$\frac{\partial}{\partial x}(v\,eq) - \frac{\partial}{\partial y}(u\,eq) = \frac{\partial \zeta_g}{\partial t}$$

Using continuity

$$\frac{\partial u_{ag}}{\partial x} + \frac{\partial v_{ag}}{\partial y} + \frac{\partial \omega}{\partial p} = 0 \quad \text{and} \quad \frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} = 0$$

QG Vorticity equation

$$\boxed{\frac{\partial \zeta_g}{\partial t} = -\vec{v}_g(\zeta_g + f_0) + f_0\left(\frac{\partial \omega}{\partial p}\right)}$$

where... $\zeta_g = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \frac{1}{f_0}\nabla^2\underline{\Phi}$

Laplacian of geopotential

Differences with full vorticity equation $\partial\zeta/\partial t$

Retained: Horizontal advection, vortex stretching

Neglected: Vertical advection, tilting, friction.

QG Thermodynamic Equation

Adiabatic form:

$$\frac{\partial T}{\partial t} = -\vec{V}_g \nabla T + \frac{\sigma p}{R} \omega = 0$$

Advection

Adiabatic correction factor for T, because not conserved with vertical motion.

$$\sigma = -\frac{RT}{P} \frac{d \ln \theta}{dp}$$

→ Static stability: Assumed to be only a function of pressure.

Diabatic:

$$\frac{\partial T}{\partial t} = -\vec{V}_g \nabla T + \frac{\sigma p}{R} \omega + \frac{J}{c_p}$$

Heating term ($\dot{Q}$)

$\dot{Q}$ sources / sinks:

- Latent heat release → most important one for synoptic scale!
- Radiation
- Friction

QG Omega Equation (Adiabatic)

Using thermodynamic equation, replace T by vertical derivative of geopotential ($\partial\Phi/\partial p$) which is related to T by hypsometric equation.

Then divide by $-P/R$, differs only by factor g_0

$$\frac{\partial}{\partial t}\left(\frac{\partial\Phi}{\partial p}\right) + \vec{V}_g \nabla\left(\frac{\partial\Phi}{\partial p}\right) + \sigma\omega = 0$$

χ = geopotential height tendency = $\partial\Phi/\partial t$

$$\boxed{\frac{\partial\chi}{\partial p} = -\vec{V}_g \nabla \frac{\partial\Phi}{\partial p} - \sigma\omega}$$ QG thermodyn equation.

Recall ..

$$\frac{\partial\zeta_g}{\partial t} = \frac{1}{f_0}\frac{\partial}{\partial t}\left(\nabla^2\Phi\right) = \frac{1}{f_0}\nabla^2\chi$$

Substitute for $\zeta_g = \frac{1}{f_0}\nabla^2\Phi$ in vorticity eqn.

QG vorticity eqn!

$$\boxed{\nabla^2\chi = -f_0 \vec{V}_g \cdot \nabla\left(\frac{1}{f_0}\nabla^2\Phi + f_0\right) + f_0^2\frac{\partial\omega}{\partial p}}$$

Steps to derive final QG omega equation from here:

- Take ∇^2 (Thermodyn. eqn) $- \frac{\partial}{\partial p}$ (Vort. eqn)
- Divide by σ

<u>Result: QG omega equation</u>

$$\boxed{\underbrace{\left[\nabla^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right]\omega}_{(A)} = \underbrace{\frac{f}{\sigma}\frac{\partial}{\partial p}\left\{\vec{V}_g \cdot \nabla\left(\frac{1}{f_0}\nabla^2\Phi + f\right)\right\}}_{(B)} + \underbrace{\frac{1}{\sigma}\nabla^2\left\{\vec{V}_g \cdot \nabla\left(\frac{-\partial\Phi}{\partial p}\right)\right\}}_{(C)}}$$

A diagnostic equation for ω that can be solved (iteratively)

<u>Physical interpretation</u>

A: 3D Laplacian of ω

B: Vertical derivative of geostrophic vorticity advection by $\vec{V}_g$ (differential vorticity advection)
Positive: PVA increasing with height
$\rightarrow$ Ascent
Negative: NVA increasing with height
$\rightarrow$ Descent

Likewise, converse if decreasing with height

Operational practice: Consider PVA, NVA at 500-mb (level of non-divergence)

C. Thermal advection

Warm Air Advection (WAA) $\rightarrow$ Ascent

Cold Air Advection (CAA) $\Rightarrow$ Descent

Operational practice! Consider at 700-mb or 850-mb since ∇T is typically larger there as compared to higher aloft.

($\dot{Q}$) Diabatic heating (Missing!!)

$\nabla^2 \dot{Q}$ Positive $\rightarrow$ Ascent

$\nabla^2 \dot{Q}$ Negative $\rightarrow$ Descent

Operational practice: Must be inferred by any information related to organized convection or potential driver thereof (satellite, radar, SSTs)

$\dot{Q}$ important over open oceans with warm water

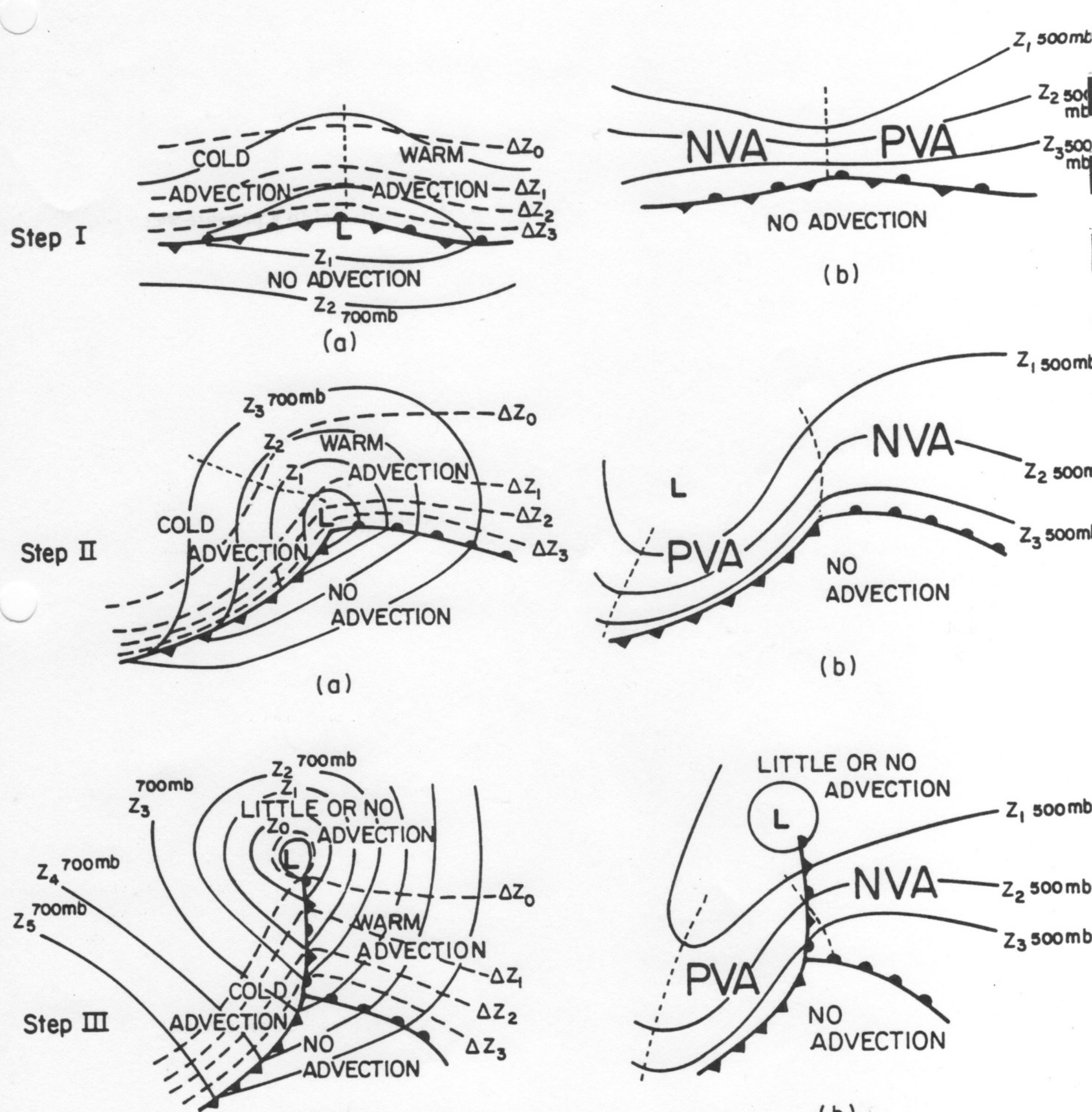


Figure 3.20: Schematic illustration of thickness advection and vorticity advection associated with extratropical cyclone development. The 700 mb height contours (solid lines), 1000-500 mb thickness contours (dashed lines), and surface frontal location are given in (a). The 500 mb height contours (solid lines) and surface frontal location are shown in (b).

Solving QG

Can be done numerically by relaxation, a standard technique to solve an elliptic equation of the form $\nabla^2\chi$ (Laplacian) More in my modeling course...

Caveats to QG

- Vertical motion is net result of all forcing terms. Terms may cancel each other out!!

- Always remember scaling assumptions! Will necessarily neglect any of the effects that lead to vertical motion on mesoscale.

- For above reason, QG vertical motion more reasonable in cool vs. warm season (where more convective)

- In practice, QG omega equation should be used to make qualitative diagnoses of vertical motion from upper-air data, that should conform to anything diagnosed in a numerical model.

Q-vectors

Motivation: We need a convenient way to diagnose QG vertical motion directly from wind and temperature fields on a Wx map.

Derivation

u-momentum equation (QG)

$$\frac{\partial u_g}{\partial t} + u_g \frac{\partial u_g}{\partial x} + v_g \frac{\partial u_g}{\partial y} - fv + \frac{\partial \Phi}{\partial x} = 0$$

$$v_{ag} + v_g = v \qquad f v_g = \partial \Phi / \partial x$$

Yields:

$$\frac{\partial u_g}{\partial t} + \vec{v}_g \nabla u_g - f v_{ag} = 0$$

Take derivative wrt z and multiply by f.

Momentum result

$$\left(\frac{\partial}{\partial t} + \vec{v}_g \nabla \right) f \frac{\partial u_g}{\partial z} - f^2 \frac{\partial v_{ag}}{\partial z} = -f \frac{\partial \vec{v}_g}{\partial z} \nabla u_g$$

RHS of equation contains thermal wind

*See aside.

$$-f \frac{\partial \vec{v}_g}{\partial z} \cdot \nabla u_g = \frac{-g}{\theta_0} (\hat{k} \times \nabla \theta) \cdot \nabla u_g$$

*equivalent to $\hat{k} \times \partial \vec{v}_g / \partial y$

$\boxed{\text{ASIDE}}$

RHS of equation contains thermal wind

$$-f \frac{\partial \vec{v}_g}{\partial z} \cdot \nabla u_g = \frac{-g}{\theta_0}\left(\hat{k} \times \nabla \theta\right) \cdot \nabla u_g$$

Expressed with θ here.

equivalent to $\hat{k} \times \partial \vec{v}_g / \partial y$

Proof

$$\vec{V}_g = u_g \hat{i} + v_g \hat{j}$$

$$\nabla u_g = \hat{i} \frac{\partial u_g}{\partial x} + \hat{j} \frac{\partial u_g}{\partial y}$$

$$\nabla v_g = \hat{i} \frac{\partial v_g}{\partial x} + \hat{j} \frac{\partial v_g}{\partial y}$$

$$\hat{k} \times \frac{\partial \vec{v}_g}{\partial y} = \hat{k} \times \left(\hat{i} \frac{\partial u_g}{\partial y} + \hat{j} \frac{\partial v_g}{\partial y} \right)$$

$$\hat{k} \times \hat{i} = \hat{j} \qquad \hat{k} \times \hat{j} = -\hat{i}$$

$$= \hat{j} \frac{\partial u_g}{\partial y} - \hat{i} \frac{\partial v_g}{\partial y}$$

since $\partial v_g / \partial y + \partial u_g / \partial x = 0$

$$= \hat{i} \frac{\partial u_g}{\partial x} + \hat{j} \frac{\partial u_g}{\partial y}$$

$$-\frac{g}{\theta_0}(\hat{k}\times\nabla\theta)\cdot\left(\hat{k}\times\frac{\partial\vec{v}_g}{\partial y}\right) = -\frac{g}{\theta_0}\frac{\partial\vec{v}_g}{\partial y}\cdot\nabla\theta$$

Final result for u-momentum

$$\boxed{\left(\frac{\partial}{\partial t}+\vec{v}_g\cdot\nabla\right)f\frac{\partial u_g}{\partial z} - f^2\frac{\partial v_{ag}}{\partial z} = -\frac{g}{\theta_0}\frac{\partial\vec{v}_g}{\partial y}\nabla\theta}$$

Thermodynamic equation (θ)

$$\frac{\partial\theta}{\partial t} + u_g\frac{\partial\theta}{\partial x} + v_g\frac{\partial\theta}{\partial y} + \frac{\theta_0}{g}N^2 w = 0$$

Boussinesq. approx. applied.

$$N^2 = g\frac{d\ln\theta_0}{dz}$$

Multiply by g/θ_0 and take derivative wrt y

$$\frac{\partial}{\partial y}\left[\left(\frac{\partial}{\partial t}+\vec{v}_g\cdot\nabla\right)\frac{g}{\theta_0}\theta + N^2 w\right] = 0$$

Final result for thermodynamic

$$\left(\frac{\partial}{\partial t}+\vec{v}_g\cdot\nabla\right)\frac{g}{\sigma}\frac{\partial\theta}{\partial y} + N^2\frac{\partial w}{\partial y} = -\frac{g}{\theta_0}\frac{\partial\vec{v}_g}{\partial y}\nabla\theta$$

Add momentum and thermodynamic results ..

First terms in each equation cancel because of thermal wind relation

$$N^2 \frac{\partial w}{\partial y} - f^2 \frac{\partial v_{ag}}{\partial z} = -2 \frac{g}{\theta_0} \frac{\partial \vec{v}_g}{\partial y} \nabla\theta$$

Q-vector in y-direction

$$\frac{-g}{\theta_0} \frac{\partial \vec{v}_g}{\partial y} \nabla\theta \, \hat{j} = Q_y$$

Q-vector in x-direction (analogously)

$$\frac{-g}{\theta_0} \frac{\partial \vec{v}_g}{\partial x} \nabla\theta \, \hat{i} = Q_x$$

Total Q-vector

$$\boxed{\vec{Q} = -\frac{g}{\theta_0} \left(\frac{\partial \vec{v}_g}{\partial x} \cdot \nabla\theta, \frac{\partial \vec{v}_g}{\partial y} \cdot \nabla\theta \right) = (Q_x, Q_y)}$$

Physical interpretation of Q-vector

Ageostrophic components of wind act to maintain thermal wind balance. An overturning circulation balances geostrophic advection of θ

$$2Q_y = N^2 \frac{\partial w}{\partial y} - f^2 \frac{\partial v_{ag}}{\partial z} = -2\frac{g}{\theta_0}\frac{\partial \vec{v}_g}{\partial y} \cdot \nabla\theta$$

↑ Measure of over turning circulation

↑ Geostrophic advection of θ.

Divergence of Q-vector defines QG.
omega equation

$$\boxed{N^2 \nabla^2 w + f^2 \frac{\partial}{\partial z}\frac{\partial(\rho w)}{\rho \partial z} = 2\nabla \cdot \vec{Q}}$$

* Negative sign if use ω as in book.

<u>In book</u>: Magnitude given by

$$\boxed{\begin{aligned} Q_1 &= -\left[\frac{\partial u_g}{\partial x}\frac{\partial \theta}{\partial x} + \frac{\partial v_g}{\partial x}\frac{\partial \theta}{\partial y}\right] \\ Q_2 &= -\left[\frac{\partial u_g}{\partial y}\frac{\partial \theta}{\partial x} + \frac{\partial v_g}{\partial y}\frac{\partial \theta}{\partial y}\right] \end{aligned}}$$

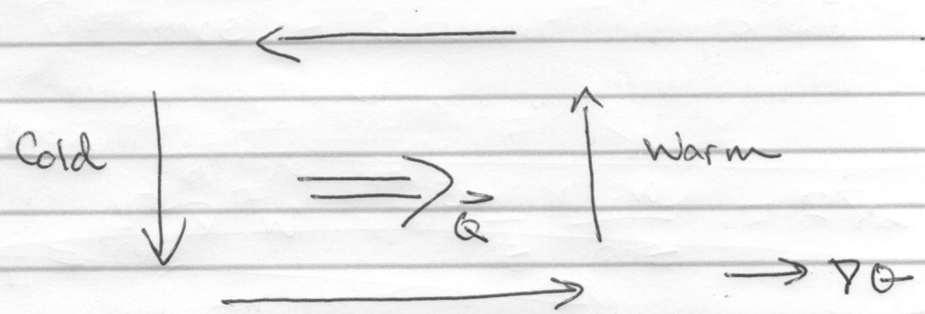
⇐ How you can use to qualitatively diagnose QG-vertical motion on Wx map. (Fig. 2.10)

$\nabla \vec{Q} = +$ $\Rightarrow$ Convergence of Q vectors
RISING AIR

$\nabla \vec{Q} = -$ $\Rightarrow$ Divergence of Q vectors
SINKING AIR

2 circumstances arise depending on which way Q-vector points

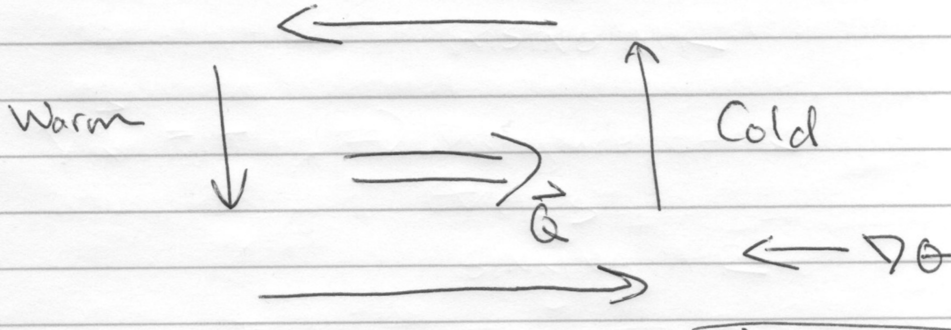
Fronto<u>genesis</u> : Thermally direct



$$\boxed{\vec{Q}\cdot\nabla\theta > 0}$$

Ageostrophic circulation tries to weaken $\nabla\theta$

Frontolysis: Thermally-indirect



$$\boxed{\vec{Q}\cdot\nabla\theta < 0}$$

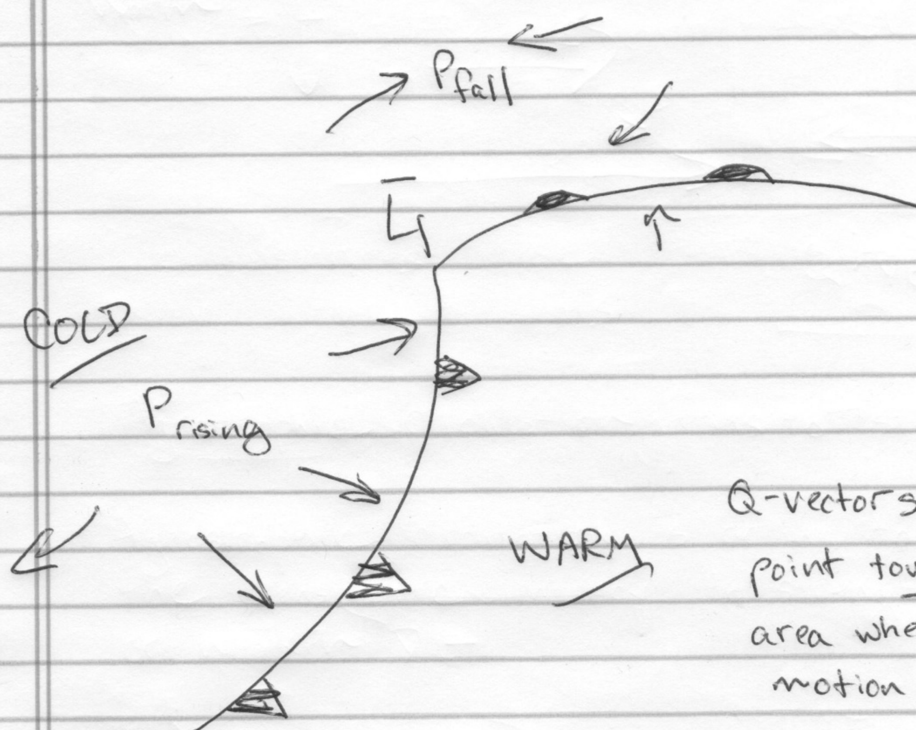
Ageostrophic circulation tries to strengthen $\nabla\theta$

Ageostrophic circulation is acting to oppose the geostrophic tendency to strengthen or weaken $\nabla\theta$.

In the case of frontogenesis, intensifying gradients mean that ageostrophic circulation is not strong enough to overcome the effects of the concentration of $\nabla\theta$ by advective terms.

Aside: Strength of fronts will be inherently underestimated by QG-theory, because effects of advection by v_{ag} are not accounted for.

Q-vectors in relation to mid-latitude cyclone



See Fig 6.13

Q-vectors will point toward area where rising motion.

Advantages of Q-vector vs. QG-ω:

1) No cancellation problem (if tendencies are opposing)
2) Q-vectors computed at single level (no differential aspect)
3) Patterns of divergence given concentrated areas with synoptic-scale vertical motion likely to occur. So better proxy for any model computed ω.