METHODS OF COMPUTING VERTICAL MOTION IN THE ATMOSPHERE [1]

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ABSTRACT

The theory of a number of different techniques for the computation of vertical motion in the atmosphere is discussed. A comparison of two independent techniques shows that both usually yield vertical velocities of the correct sign and order of magnitude.

1. Introduction

Widespread deterioration and improvement of weather are usually caused by small vertical velocities of the same sign prevailing over large areas. Adiabatic cooling due to upward motion leads to the formation of clouds and precipitation, and adiabatic warming due to downward motion leads to their dissipation.

Superimposed on the widespread field of uniform vertical motion are small-scale eddies with relatively strong upward and downward currents, but the large-scale vertical velocities in question are of the order of only one or two centimeters per second. Due to their smallness, they have not been measured directly as yet, and their direct determination in the near future is doubtful. It is possible, however, to compute vertical velocities from observed upper-air data. Two independent methods have been proposed for this purpose, the *adiabatic* and the *kinematic* methods. The first is based on the assumption that changes of state of atmospheric air are adiabatic, and the second depends on the principle of conservation of mass as expressed by the equation of continuity. For each of these methods of computing vertical velocities, several techniques have been used. These are outlined below.

2. The adiabatic method

Advective technique.—Using the rules of calculus one may expand the individual temperature change into

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + \mathbf{V}\cdot\nabla T + w\frac{\partial T}{\partial z} \tag{1}$$

where T is temperature, t time, $\mathbf{V}$ the horizontal velocity vector, ∇ the vector differential operator applied in the horizontal direction, z the vertical coordinate, and w the vertical velocity. Since

$$\frac{dT}{dt} = \frac{dT}{dz}\frac{dz}{dt} = w\frac{dT}{dz},$$

equation (1) may be written

$$\frac{\partial T}{\partial t} + \mathbf{V}\cdot\nabla T + w\left(\frac{\partial T}{\partial z} - \frac{dT}{dz}\right) = 0. \tag{2}$$

From the first law of thermodynamics, for adiabatic, reversible changes of state of a perfect gas,

$$c_p dT = \frac{1}{\rho}dp$$

where p is pressure, ρ density, and c_p the specific heat at constant pressure. Multiplication of both sides of this equation by w/dz gives

$$wc_p\frac{dT}{dz} = \frac{w}{\rho}\frac{dp}{dz}.$$

Now, $w\,dp/dz$ is nearly equal to $w\,\partial p/\partial z$,[2] which is given by $-g\rho w$, according to the hydrostatic equation, where g is the acceleration of gravity. Thus,

$$-w\frac{dT}{dz} = w\frac{g}{c_p} = w\gamma_{ad}$$

where γ_{ad} is the dry adiabatic lapse rate. Hence, equation (2) may be written

$$\frac{\partial T}{\partial t} + \mathbf{V}\cdot\nabla T + w(\gamma_{ad} - \gamma) = 0, \tag{3}$$

where γ is the existing lapse rate. If the air is saturated and rising, γ_{ad} should be interpreted as the saturation adiabatic lapse rate.

[1] An expansion of part of a progress report on a research project conducted at New York University and sponsored by the Army Air Forces Weather Service.

[2] Actually, $w\,dp/dz = w\,\partial p/\partial z + \mathbf{V}\cdot\nabla p + \partial p/\partial t$. The first term on the right is of the order of 1 dyne cm^{-2} sec^{-1}. The sum of the last two terms is about ten times smaller, and may usually be omitted. According to observation, if $\partial p/\partial t$ is unusually large, $w\,\partial p/\partial z$ also is unusually large, and to put $w\,dp/dz = w\,\partial p/\partial z$ is still permissible.

With the exception of w, all terms in equation (3) can be measured; therefore, equation (3) can be used to compute w. The local temperature change is computed from a graph showing the observed local temperature at a given level as a function of time. The advective term, $\mathbf{V} \cdot \nabla T$, is found on a constant-level or constant-pressure chart by measuring the temperature gradient and multiplying it by the wind component parallel to the temperature gradient.

Single-station technique.—The single-station technique, suggested by Panofsky (4), also utilizes equation (3). It differs from the advective technique in that the advective term is computed from a hodograph of upper-air winds. Thus, all terms in equation (3) can be measured at a single station. This advantage is offset by two disadvantages: (a) Observed winds have to be used; thus the computations are limited to good weather, unless radio wind observations are available. (b) The theory of the determination of advection from a hodograph is based on the assumption of geostrophic wind; in spite of this assumption, however, vertical velocities computed by the single-station technique have been found to be well correlated with weather changes.

Isobaric technique.—The isobaric technique has been described by Miller.[3] It is similar to techniques suggested by Klyucharev (3) and Petterssen.[4]

Let $\delta T/\delta t$ denote the temperature change per unit time following a point which has the same horizontal component of motion, $\mathbf{V}$, as the air, but moves on a constant pressure surface. From the expansion of $\delta T/\delta t$, it follows that

$$\frac{\delta T}{\delta t} = \frac{\partial T}{\partial t} + \mathbf{V} \cdot \nabla T + w_p \frac{\partial T}{\partial z} \tag{4}$$

where w_p is the vertical velocity of the moving point. This is usually about one-tenth as large as the other terms in this equation and may be neglected. On adding equations (3) and (4), one gets

$$\frac{\delta T}{\delta t} = -w(\gamma_{ad} - \gamma). \tag{5}$$

The quantity $\delta T/\delta t$ is measured by subtracting the temperature at the beginning from that at the end of a 12-hour trajectory on an isobaric surface. The value of $\gamma_{ad} - \gamma$ is obtained from charts showing isopleths of observed lapse rates.

Graphical technique.—The graphical technique was suggested by Miller.[5] Using the data on certain isobaric charts (for example, the 900-, 800-, and 700-mb charts), trajectories centered at a selected geographical point are constructed on each chart to represent the 12-hour period between upper-air soundings. At the initial and final end points of a set of trajectories, "synthetic" soundings are constructed representing the variation of temperature with pressure, as do the usual soundings. However, the temperatures are not directly observed but are obtained from an analysis of the isotherms on the isobaric charts, and the synthetic soundings are not vertical soundings because of the effect of vertical wind shear.

If there had been no vertical motion, the initial and final soundings would be identical. Actually, they will not be identical, and the assumption of adiabatic vertical motion permits the estimate of how much vertical motion at various levels was necessary to transform the initial into the final sounding.

The graphical technique has the advantage of showing most clearly the effect of the vertical motion, and it permits easy evaluation of vertical motion in case of saturated air. It has the disadvantage that it requires more labor than some of the other techniques.

Isentropic technique.—The isentropic technique was developed by Fleagle.[6] It makes use of the fact that, if processes are dry-adiabatic, the potential temperature is conserved and the isentropic surfaces follow the three-dimensional motion. Two isentropic surfaces for the same potential temperature are constructed, 12 hours apart, and air trajectories for the period between the two charts are drawn. The height at the beginning of the trajectory is read from the first chart, and that at the end of the trajectory from the second chart. The difference in height measures the net vertical motion in the 12 hours.

The isentropic technique is fastest if isentropic data are transmitted, but with the data transmitted at present it is somewhat cumbersome. A further disadvantage lies in the fact that the technique cannot be modified easily to take care of saturated air.

Critique of adiabatic techniques.—All adiabatic techniques give inaccurate results when the stability is small. All adiabatic techniques use wind and radiosonde data; with the exception of the single-station technique the winds can be, if necessary, computed from the pressure field, and thus these techniques can be used in bad weather and at high levels. Since all adiabatic techniques are based on the assumption of adiabatic changes, they should not be used to compute vertical velocities in the lower layers of the atmosphere, where non-adiabatic processes have a pronounced effect.

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[3] R. G. Fleagle, H. A. Panofsky, H. T. Mantis, and J. E. Miller, "On vertical motion in the atmosphere," Army Air Forces Weather Service, Technical Report 105-3, Langley Field, Va. In preparation.

[4] S. Petterssen, "An investigation of subsidence in the free atmosphere," U. S. Navy, Aerology Section, 1944.

[5] *loc. cit.*[3]

[6] *loc. cit.*[3]

3. The kinematic method

The equation of continuity can be written in the form

$$\frac{\partial w}{\partial z} + \frac{1}{\rho}\frac{d\rho}{dt} = -\nabla \cdot \mathrm{V}. \tag{6}$$

Now

$$\frac{1}{\rho}\frac{d\rho}{dt} = \frac{1}{\rho}\frac{d\rho}{dz}\frac{dz}{dt} = \frac{w}{\rho}\frac{d\rho}{dz}.$$

If density changes are adiabatic,

$$\frac{w}{\rho}\frac{d\rho}{dz} = \frac{w}{\Gamma p}\frac{dp}{dz} \approx \frac{w}{\Gamma p}\frac{\partial p}{\partial z}$$

where Γ is the ratio of the specific heat at constant pressure to that at constant volume.[7] Thus equation (6) may be written

$$\frac{\partial w}{\partial z} + \frac{w}{\Gamma p}\frac{\partial p}{\partial z} = -\nabla \cdot \mathrm{V}. \tag{7}$$

This may be integrated from the surface (subscript s) to an arbitrary level (subscript h). The left side of the equation becomes an exact differential after multiplication by $p^{1/\Gamma}$. Integration gives

$$w_h = \left(\frac{p_s}{p_h}\right)^{1/\Gamma} w_s - \int_s^h \left(\frac{p}{p_h}\right)^{1/\Gamma} \nabla \cdot \mathrm{V}\, dz. \tag{8}$$

The value of w_s can be estimated from the slope of the terrain and the wind near the ground. The integral can be approximated by measuring the integrand at different levels and adding the results.

Wind-component technique.—The divergence is computed from the formula

$$\nabla \cdot \mathrm{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - \frac{v}{R}\tan\varphi$$

in which u and v are components of the velocity in the standard x direction (toward east) and y direction (toward north) respectively, R is the radius of the earth, and φ is latitude. The term $-(v/R)\tan\varphi$ has to be applied due to convergence of the meridians. The gradients of u and v are measured from isopleths of u and v.

Wind streamline technique.—In this technique, the divergence is measured as suggested by V. Bjerknes (2). The divergence is expressed in components in "natural coordinates," parallel and at right angles to the streamlines. These form a polar coordinate system, with its origin at the intersection of tangents to adjacent streamlines. In these coordinates

$$\nabla \cdot \mathrm{V} = \frac{\partial v}{\partial r} + \frac{v}{r}$$

where r is the distance from the origin and v is the speed, reckoned positive if directed along the positive r direction.

The first term on the right is obtained by measurement on charts showing streamlines and lines of constant speed. The second term can be computed from the same two sets of lines with the aid of overlays developed for this purpose by V. Bjerknes.

Resultant-component and resultant-streamline techniques.—Consider the identity

$$\nabla \cdot \int_s^h \left(\frac{p}{p_h}\right)^{1/\Gamma} \mathrm{V}\, dz \equiv \frac{\partial}{\partial x}\int_s^h \left(\frac{p}{p_h}\right)^{1/\Gamma} u\, dz + \frac{\partial}{\partial y}\int_s^h \left(\frac{p}{p_h}\right)^{1/\Gamma} v\, dz - \int_s^h \left(\frac{p}{p_h}\right)^{1/\Gamma} \frac{v}{R}\tan\varphi\, dz.$$

In accordance with the rules relating to the interchange of integration and differentiation, the identity may be written[8]

$$\begin{aligned}\nabla \cdot \int_s^h \left(\frac{p}{p_h}\right)^{1/\Gamma} \mathrm{V}\, dz = {} & \int_s^h \frac{\partial}{\partial x}\left[\left(\frac{p}{p_h}\right)^{1/\Gamma} u\right] dz + u_h\frac{\partial h}{\partial x} - u_s\left(\frac{p_s}{p_h}\right)^{1/\Gamma}\frac{\partial s}{\partial x} \\ & + \int_s^h \frac{\partial}{\partial y}\left[\left(\frac{p}{p_h}\right)^{1/\Gamma} v\right] dz + v_h\frac{\partial h}{\partial y} \\ & - v_s\left(\frac{p_s}{p_h}\right)^{1/\Gamma}\frac{\partial s}{\partial y} - \int_s^h \left(\frac{p}{p_h}\right)^{1/\Gamma}\frac{v}{R}\tan\varphi\, dz. \end{aligned} \tag{9}$$

If the upper boundary of the layer considered is horizontal, $\partial h/\partial x = \partial h/\partial y = 0$. Also $u_s\partial s/\partial x + v_s\partial s/\partial y = w_s$, the forced vertical velocity at the ground. Thus, if the terms are regrouped and the horizontal variation of $(p/p_h)^{1/\Gamma}$ is neglected, equation (9) reduces to

$$\nabla \cdot \int_s^h \left(\frac{p}{p_h}\right)^{1/\Gamma} \mathrm{V}\, dz = \int_s^h \left(\frac{p}{p_h}\right)^{1/\Gamma} \nabla \cdot \mathrm{V}\, dz - w_s\left(\frac{p_s}{p_h}\right)^{1/\Gamma}. \tag{10}$$

On combination, equations (10) and (8) become

$$w_h = -\nabla \cdot \int_s^h \left(\frac{p}{p_h}\right)^{1/\Gamma} \mathrm{V}\, dz. \tag{11}$$

Equation (11) may be rewritten in the form

$$w_h = -\left(\frac{\bar{p}}{p_h}\right)^{1/\Gamma} \nabla \cdot \int_s^h \mathrm{V}\, dz \tag{12}$$

where $\bar{p}$ is a mean pressure in the layer from the surface to the level h. Theoretically, this mean should be weighted by the wind velocity. In practice, however,

[7] *Cf.* footnote 2.

[8] See, for example, I. S. and E. S. Sokolnikoff, *Higher mathematics for engineers and physicists*, 2nd edition, pp. 167–9, McGraw-Hill Book Co., 1941.

very little error is made if $\bar{p}$ is taken as the pressure half way between the surface and the level h.

The quantity

$$\int_s^h \mathbf{V}\,dz$$

can be computed from the horizontal component of the vector connecting the starting point of a pilot-balloon run with the balloon when it reaches the level h. This component is given by

$$\mathbf{R} = \int_{t_s}^{t_h} \mathbf{V}\,dt$$

where the balloon is assumed to be released at t_s and to reach the level h at time t_h. Now,

$$\int_s^h \mathbf{V}\,dz = \int_{t_s}^{t_h} \mathbf{V}\frac{dz}{dt}\,dt,$$

or, if the rate of ascent of the balloon, dz/dt, is assumed approximately constant and equal to b, then

$$\int_s^h \mathbf{V}\,dz = b\int_{t_s}^{t_h} \mathbf{V}\,dt = b\mathbf{R}.$$

This quantity can easily be computed from the original plot of the pilot-balloon observation. The divergence of the vector can be obtained by either of the two techniques described above for obtaining the divergence of the wind.

The resultant techniques are probably much more accurate than the wind techniques, for in the first two kinematic techniques the wind velocity, which is obtained by *differentiation* of the pilot-balloon run with respect to height, is *integrated* again with respect to height. In the resultant techniques both of these needless and error-producing steps are omitted. Also, the forced surface vertical velocity does not have to be computed in the resultant techniques.

Critique of kinematic techniques.—All kinematic techniques yield instantaneous vertical velocities. All kinematic techniques use winds only; observed winds have to be used, since the divergence of the gradient and geostrophic winds usually is too small in absolute value, which means that the kinematic techniques are limited to areas of good weather until a dense network of radio wind stations is established. Wind coverage sufficient to measure divergence usually does not reach above 10,000 feet at the present time. Thus, the kinematic techniques can rarely be used above 3 km.

4. Comparison of kinematic and adiabatic methods

The kinematic and the adiabatic methods for the computation of vertical velocities are virtually independent. Although both depend on the wind, they are affected by an error in the wind in different ways. In the adiabatic method the wind itself is used to compute the path of an air particle, and in the kinematic method the space derivatives of the wind are used. In general, a quantity is not correlated with its derivative, and therefore the error of a quantity is not correlated with the error of its derivative. Hence a comparison of vertical velocities computed by the two methods at the same points should determine, to some extent, whether either method individually is capable of yielding correct vertical velocities. The isentropic and resultant-component techniques were chosen for the comparison of the two methods.

One of the difficulties encountered in this comparison stems from the fact that the isentropic technique yields a field of vertical velocities on a sloping surface, whereas the resultant-component technique yields a field on a horizontal surface. In order to minimize the number of isentropic surfaces to be analyzed for comparison with a single horizontal surface, it was assumed that a vertical velocity computed at a given point on an isentropic surface applied also within 500 m above or below that surface.

The vertical velocities computed by the isentropic technique were based on the height change of air parcels along 12-hour trajectories. Thus they represent values averaged in time as well as in space.

Since the adiabatic method is not applicable at low levels, and sufficient data are not available for the use of the kinematic method much above 3 km, the comparison of the two methods is limited to elevations between about 2 and 3 km. In this comparison most points were located at an elevation of 2070 m, with a few points at 3050 m.

Due to the necessity of using observed winds in the kinematic method, the area of comparison is restricted to regions of good weather. Hence, it is to be expected that downward vertical velocities will be preponderant in the comparison.

It was found that the computation of divergence from change of resultants in 100 or 200 km yielded erratic values, probably because local turbulence and errors in the wind observations produced local variations of the same magnitude as the large-scale divergence to be measured. Therefore, gradients were measured over 500 km and averaged within a square 500 km on the side. Observed wind resultants at 1000 or 2200 GMT were used, and the vertical velocities at the two levels were computed from equation (12) as described above. Thus, the vertical velocities computed by the resultant-component technique were instantaneous values averaged over a 500-km square.

Figure 1 shows the comparison of the vertical velocities computed by the two techniques. For the periods 1000 GMT 1 December to 2200 GMT 3 December 1944, and 1000 GMT 31 October to 2200 GMT 3 November 1944, vertical velocities were com-

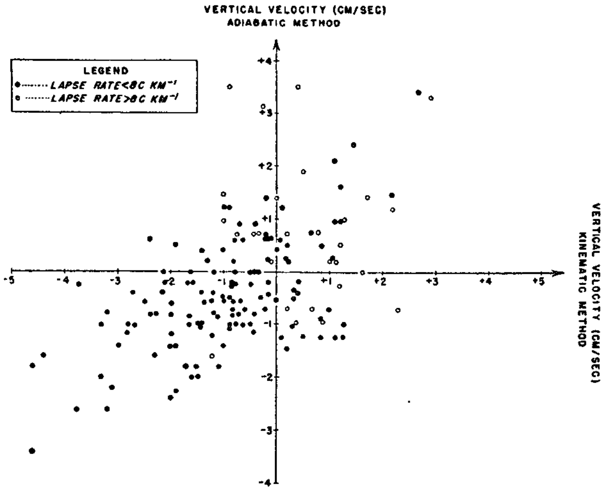

FIG. 1. Comparison of vertical velocities computed by the adiabatic and kinematic methods.

puted at the two levels mentioned above, wherever sufficient data were available. Vertical velocities computed in layers of lapse rates 8 C km^{-1} or greater are indicated by open circles in Figure 1, since in these layers the vertical velocities computed by the adiabatic method are very likely to be inaccurate.

The figure shows that both methods generally yield vertical velocities of the same sign and order of magnitude. This may be regarded as an indication that each method will generally furnish vertical velocities of the right sign and order of magnitude, but that it may fail to do so in particular cases.[9]

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[9] Additional proof that the adiabatic method usually yields vertical velocities of the right sign is found in the comparison of vertical velocities with weather changes, published elsewhere (footnote 3).