Modal and Non-Modal Baroclinic Waves

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ABSTRACT

Solution of the initial-value problem for the Eady model is presented. In the presence of boundaries, normal mode waves as well as non-modal waves exist. Energy extracted from the mean flow during the initial development of a perturbation is found to excite the persistent normal modes. It is suggested that this process may be important to cyclogenesis and in providing energy to neutral or near-neutral normal modes. In particular, the Petterssen criterion for cyclogenesis is clarified.

1. Introduction

Understanding of the process by which the potential energy of vertically sheared geostrophic mean flows is tapped to provide energy to growing synoptic-scale waves is based on the quasi-geostrophic instability theory of Charney (1947) and Eady (1949). The equation expressing conservation of pseudo-potential vorticity in perturbation form results from this development and, together with attendant boundary conditions, may be regarded as an eigenvalue problem with the vertical structure of the normal mode as the eigenfunction and the phase speed c as the eigenvalue. Complex values of c are found for exponentially growing and decaying modes and the stability of the flow is implicitly associated with the existence of these modes. As has been noted previously, however (Pedlosky, 1964), there are a number of subtleties involved in this approach. In the event that there exists a complete set of normal modes for a given problem so that an arbitrary initial condition can be represented by a combination of these, the procedure is well-founded, but this state of affairs rarely occurs in practice. More commonly, there are a finite number of discrete normal modes that must be augmented by a continuous spectrum to obtain a complete set. In either case, the existence of one growing normal mode assures the instability of the flow in that an arbitrarily small excitation of this mode will eventually give rise to an unbounded disturbance. The stability of the flow can be assured only by the existence of a complete set of decaying modes.

For the case of plasma instabilities where the normal mode is excited by thermal noise and 10 e-foldings may be required to make it comparable with mean quantities, the existence of an exponential mode is probably necessary to produce an observable disturbance. In the case of rapid and lee cyclogenesis typical deepenings are ~ 2 e-foldings [Buzzi and Tibaldi, (1978); see their Fig. 12] so that the existence of an exponential mode may be important only late in the evolution of an initial state. During the early stage of growth, the continuous spectrum is equally involved in the interaction with the mean flow [Farrell (1982), hereafter F1].

This work is concerned with exploring the connection between the discrete normal modes and the continuous spectrum in the early development of disturbances in baroclinic flows which provides an explanation for the rapid growth of surface depressions arising from certain initial conditions, as, for example, when the depression is overtaken by a short-wave upper trough (Petterssen, 1955).

To be explicit, if the variation in the mean flow is confined to a vertical shear and it is assumed that Fourier decomposition is valid in the zonal direction, then there is a distinction to be drawn between the modal response in the form

$$\psi(x, z, t) = \psi(z)F(t)e^{ikx},$$

where ψ is the perturbation geostrophic streamfunction, x eastward distance, z height, k zonal wavenumber, t time; and the nonmodal response

$$\psi(x, z, t) = \psi(z, t)e^{ikx}.$$

The modal amplitude variation may be exponential, of the form

$$F(t) = e^{\sigma_r t}e^{i\sigma_i t},$$

as in the Charney mode and Burger/Green modes of the Charney problem, or algebraic

$$F(t) = t^{\alpha}e^{i\sigma_i t}$$

as at the neutral point of the Charney problem (Burger 1966).

The non-modal time dependences are typically a complicated mixture of algebraic modulations in am-

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plitude of an oscillatory signal of variable frequency and vertical wavelength. Examples include the Couette solution of Orr (1907), the internal wave in linear shear (Phillips, 1966), and Rossby waves in linear shear (Yamagata, 1976). In these cases, the energy of the perturbation grows as long as the orientation of the disturbances is such as to give rise to Reynolds stresses which negatively correlate with the shear; when the Reynolds stress and shear become positively correlated the perturbations decay asymptotically. As originally pointed out by Orr (1907), this ultimate decay which must be the result of any $t \longrightarrow \infty$ asymptotic analysis of these problems comes only after what may be a very great increase in amplitude.

It is important to notice that these exact solutions were obtained for problems which do not support normal modes. When such are present, energy extracted from the mean during the initial growth may be projected on the modal disturbances whether these are unstable or not (see F1). This qualitatively changes the long-time behavior from decay to persistence if the modes are neutral, as was noted in a Fourier/Laplace asymptotic analysis of the Eady problem (Pedlosky, 1964).

In the following section the relation between modal and non-modal solutions is examined to discover the way in which normal modes arise from initial perturbations of plane-wave form. This last implies no loss of generality as an arbitrary perturbation may be obtained from these by Fourier synthesis.

2. Formulation of the Eady problem

Perhaps the simplest example illustrating these ideas is the model of Eady (1949), for which the shear and static stability are taken to be constant, the planetary vorticity gradient ignored, and the Boussinesq approximation made, resulting in the perturbation potential vorticity equation (Pedlosky, 1979)

$$\left(\frac{\partial}{\partial \tilde{t}} + i\tilde{\alpha}\tilde{z}\right)(\psi_{\tilde{z}\tilde{z}} - \tilde{\alpha}^2\psi) = 0, \tag{1}$$

where

$$\left.\begin{aligned} \psi &= \psi(\tilde{z}, \tilde{t})e^{i(\tilde{k}\tilde{x}+\tilde{l}\tilde{y})} \\ \tilde{\alpha} &= \sqrt{\epsilon}H(\tilde{k}^2 + \tilde{l}^2)^{1/2} \end{aligned}\right\}.$$

The following nondimensionalizations have been made:

$$\left.\begin{aligned} \tilde{t} &= \frac{t\Lambda\sqrt{\epsilon}k}{\alpha} \\ \tilde{k} &= kH/\sqrt{\epsilon} \\ \tilde{z} &= z/H \end{aligned}\right\},$$

where $\epsilon = f^2/N^2$ is the square ratio of the Coriolis parameter to the Brunt Väisälä frequency, H height of upper boundary and Λ the vertical shear. The boundary conditions express the vanishing of the vertical velocity at horizontal boundaries:

$$\left(\frac{\partial}{\partial \tilde{t}} + i\tilde{\alpha}\tilde{z}\right)\psi_{\tilde{z}} - i\tilde{\alpha}\psi = 0. \tag{2}$$

Tildes are dropped in the sequel.

3. The Eady edge wave

To begin we will relax the upper boundary condition (replacing H by an arbitrary scale of, say, 10 km), requiring only that the streamfunction remain bounded as $z \longrightarrow \infty$. The physical situation is that of a semi-infinite sheared ocean on an f-plane.

The method of solution is as in Orr (1907), and the notation follows Simmons and Hoskins (1979). Making use of the fact that interior equation (1) has a particular solution for the plane wave initial condition,

$$\psi(x, y, z, 0) = e^{i(kx+ly+mz)}$$

of

$$\psi_p(x, y, z, t) = \frac{(1 + a^2)}{[1 + (a - t)^2]} e^{i(kx+ly+(m-\alpha t)z)},$$

where

$$a = \frac{m}{\alpha}$$

as well as the bounded homogeneous solution

$$\psi_h = e^{i(kx+ly)}e^{-\alpha z}.$$

The general solution

$$\psi = \psi_p + A(t)\psi_h$$

must satisfy (2) which requires

$$\frac{dA}{dt} + iA = -if(t),$$

$$f(t) \equiv \frac{2(1 + a^2)}{(1 + (a - t)^2)^2}.$$

This is solved for $A(t)$ subject to the initial condition $A(0) = 0$ by

$$A(t) = -ie^{-it}\int_0^t f(\tau)e^{i\tau}d\tau.$$

While the non-modal contribution ψ_p approaches zero asymptotically as t^{-2} after obtaining a maximum of $(1 + a^2)$ at $t = a$, the modal part ψ_h tends to a nonzero limit independent of the shear

$$A(\infty) = -ie^{-it}\int_0^\infty f(\tau)e^{i\tau}d\tau,$$

which is large if a is large. This example shows the way in which an initial non-modal perturbation produces a persistent modal disturbance. It is not most simply understood as a projection of an initial condition on a normal mode, as there is no orthogonality between the modes to make this concept useful heuristically although it is formally correct (F1).

4. Eady initial value problem

Placing a lid at $Z = H$ in the previous example results in the familiar Eady problem which supports both neutral and exponential normal modes as well as a continuum of singular modes completing the spectrum. Laplace transform asymptotics for this problem (Pedlosky, 1964) show that the solution is dominated by the exponentially growing mode where there is one, but that the asymptotic amplitude where the normal modes are neutral is inextricably linked with the continuous spectrum. Our analysis begins by taking account of the boundary by setting the vertical scale to H, obtaining from (1) and (2)

$$\left(\frac{\partial}{\partial t} + i\alpha z\right)(\psi_{zz} - \alpha^2\psi) = 0, \quad (3)$$

$$\frac{\partial^2\psi}{\partial t\partial z} - i\alpha\psi = 0, \qquad z = 0, \quad (4a)$$

$$\frac{\partial^2\psi}{\partial t\partial z} + i\alpha\frac{\partial\psi}{\partial z} - i\alpha\psi = 0, \quad z = 1. \quad (4b)$$

Again (3) has the particular solution for the plane wave initial condition given in Section 3, but the homogeneous solution growing with Z must now be retained. The total streamfunction is of the form

$$\psi = \left\{\frac{(1 + a^2)}{[1 + (a - t)^2]} e^{i(m-\alpha t)z} + A(t)\cosh(\alpha z) + B(t)\sinh(\alpha z)\right\} e^{i(kx+ly)}.$$

Requiring this to satisfy the boundary conditions (4a), (4b) results in the simultaneous equations

$$\frac{dB}{dt} - iA = if,$$

$$\frac{dA}{dt} + \coth(\alpha)\frac{dB}{dt} + i(\alpha - \coth(\alpha))A + i[\alpha\coth(\alpha) - 1]B = \frac{ie^{i\alpha(a-t)}}{\sinh(\alpha)}f(t).$$

Subject to the initial condition $A(0) = B(0) = 0$, the solution is

$$A = \frac{1}{\sigma_1 - \sigma_2}[e^{\sigma_1 t}L_i(t) - e^{\sigma_2 t}L_2(t)], \quad (5a)$$

$$B = \frac{i}{\sigma_1 - \sigma_2}\left[\frac{e^{\sigma_1 t}}{\sigma_1}L_1(t) - \frac{e^{\sigma_2 t}}{\sigma_2}L_2(t)\right], \quad (5b)$$

where

$$\left.\begin{aligned} L_1(t) &\equiv g_{-1}(t)[-i\sigma_1\coth(\alpha) + \alpha\coth(\alpha) - 1] + g_{+2}(t)\frac{i\sigma_1 e^{im}}{\sinh(\alpha)} \\ L_2(t) &\equiv g_{-2}(t)[-i\sigma_2\coth(\alpha) + \alpha\coth(\alpha) - 1] + g_{+1}(t)\frac{i\sigma_2 e^{im}}{\sinh(\alpha)} \\ g_{+i}(t) &\equiv \int_0^t f(\tau)e^{\sigma_i\tau}d\tau \\ g_{-i}(t) &\equiv \int_0^t f(\tau)e^{-\sigma_i\tau}d\tau \end{aligned}\right\}.$$

The eigenvalues associated with the normal modes are

$$\left.\begin{aligned} \sigma_1 &= \frac{-i\alpha}{2} + \left[\left(\frac{\alpha}{2} - \tanh\frac{\alpha}{2}\right)\left(\coth\frac{\alpha}{2} - \frac{\alpha}{2}\right)\right]^{1/2} \\ \sigma_2 &= \frac{-i\alpha}{2} - \left[\left(\frac{\alpha}{2} - \tanh\frac{\alpha}{2}\right)\left(\coth\frac{\alpha}{2} - \frac{\alpha}{2}\right)\right]^{1/2} \end{aligned}\right\}.$$

The normal modes are set up on the nondimensional time scale of $f(t)$ which is $a = m/\alpha$. This is the time scale of the non-modal waves. There is another time scale which arises from the interference of the normal modes and the resulting interaction with the mean (Lindzen *et al.*, 1982); the period of which is $\tau = 2\pi(\mathrm{Im})\sigma_1 - \mathrm{Im}\sigma_2)^{-1}$. As a consequence of this interaction, large-amplitude asymptotic neutral modes result for values of α slightly greater than the unstable/neutral transition wavenumber α_c, which satisfies

$$\tfrac{1}{2}\alpha_c = \coth\tfrac{1}{2}\alpha_c.$$

This can be seen from (5) and the fact that the eigenvalues coalesce at this point. Taking the limit of (5) at α_c reveals that the asymptotic amplitude grows linearly with time as expected for coalescing poles in Laplace transform theory, i.e.,

$$\left.\begin{aligned} A(t; \alpha_c) &\approx te^{-i\alpha_c t/2}L(t; \alpha_c) \\ B(t; \alpha_c) &\approx -te^{-i\alpha_c t/2}L(t; \alpha_c)2\alpha_c^{-1} \end{aligned}\right\}.$$

The implication that "nearly resonant" growth found in the vicinity of coalescing eigenvalues in initial value problems is of importance in geophysical flows is encouraged by the similar coalescence at the neutral point in the Charney problem (Burger, 1966). As was suggested in F1, the excitation of neutral and/or slightly unstable modes by this mechanism may be equally as important as exponential instability in accounting for observed wave spectra.

5. Relation to cyclogenesis

Rapid development of a surface depression is often associated with its being "overtaken" by a higher level trough (Petterssen, 1955; Petterssen and Smebye, 1971). It has been shown here that the dynamics of establishing a persistent modal wave from such a non-modal initial state involves the cooperative interaction of the modal and non-modal waves with the mean flow. A single neutral normal mode, which supports no heat flux, cannot by itself extract energy from the mean flow. On the other hand, the non-modal waves alone do interact with the mean as may two or more neutral modes simultaneously present. The remarkable thing is that together the modal and non-modal waves form a complete solution which results in an arbitrarily large amount of energy extracted from the mean and deposited in the neutral modes (for appropriate initial conditions). This mechanism is not limited to neutral modes of course and examples of rapid initial development of unstable modes have been shown (F1). In order to underline the role of the initial growth, a problem is chosen here for which initially the large components of the solution are neutral. This gives rise to a rapid primary deepening resulting from the initial growth followed much later by the weak secondary growth surge of the initially small exponentially unstable mode component. Choosing a problem with highly unstable components would result in the familiar two-stage deepening. In any case the deepening cannot be regarded as a simple superposition of pressure tendencies.

The synoptic situation conducive to rapid development can be modeled by Fourier synthesizing a local disturbance of the form

$$\psi = h(x)e^{i(kx+mz-\pi/2)}\sin\left(\frac{\pi y}{L}\right), \tag{6}$$

where $h(x)$ is a split cosine bell:

$$h(x) = \begin{cases} \dfrac{1-\cos\left(\dfrac{4\pi x}{d}\right)}{2}, & 0 \leq x < \dfrac{d}{4} \\ 1, & \dfrac{d}{4} \leq x \leq \dfrac{3d}{4} \\ \dfrac{1-\cos\left[\dfrac{4\pi(x-d/2)}{d}\right]}{2}, & \dfrac{3d}{4} < x \leq d. \end{cases} \tag{7}$$

The domain is chosen to be of nondimensional extent $D = 6\pi$ in the zonal direction and $L = 1.5$ in width. The initial disturbance has wavenumber $k = 3.0$ with associated wavelength $\lambda = 2\pi/3$ and it is confined in x to two wavelengths [$d = 2\lambda$ in (7)]. In all, 128 wavenumbers are included in the Fourier representations of (6).

In the first example a vertical wavenumber $m = \pi$ is chosen, corresponding to an upper-level low one-half wavelength upstream of its lower-level counterpart. The synoptic situation for typical midlatitude parameters $H = 10$ km, $\epsilon = 10^{-4}$, $\Lambda = 3$ m s^{-1} km^{-1}, is dimensionally $D = 19\,000$ km, $L = 1500$ km, $\lambda = 2100$ km.

The real part of ψ is plotted in Fig. 1. The contour interval is 1.0 and the initial condition is outlined by the zero contour which is suppressed in the subsequent figures. These are at $t = 1.0$ (Fig. 1b) $t = 2.0$ (Fig. 1c) and $t = 4.0$ (Fig. 1d), corresponding to dimensional times of 9.3, 18.6 and 37.2 h, respectively. During the 37.2 h that the isolated surface disturbance was established, the fastest growing exponential normal mode e-folded only 0.27 times and played an insignificant role in the development.

By comparison, a barotropic initial condition, $m = 0.0$, results in negligible deepening; Fig. 2a shows the initial disturbance outlined by the zero contour, while Fig. 2b is taken at $t = 4.0$, again with the zero contours suppressed.

Some remarks on similarities between this model and observations of cyclogenesis follow:

1) The Petterssen criterion which says that an upper level trough overtaking a surface depression often results in rapid deepening is understood in terms of the initial-value problem.

2) The disturbance is localized zonally and concentrated near the surface as are observed cyclones.

3) Early stages of deepening proceed much more rapidly than normal mode e-folding rates as is commonly observed.

4) Development enters a second regime at $t > a$ which roughly marks the transition from non-modal growth to exponential growth. This two-stage deepening has often been remarked on in studies of cyclogenesis (Palmen and Newton 1969; Hage 1961; Chung *et al.* 1976, Buzzi and Tibaldi, 1978).

5) Disturbance structure becomes nearly barotropic at $t \approx 1.0$ being composed primarily of neutral barotropic waves and assumes baroclinic phase tilts only as the normal mode of fastest growth becomes dominant. The time scale for this is a few e-foldings of the most unstable wave.

6) The wavelength of the cyclogenesis is determined by the dominant wave in the initial state and by the preferred scale of initial growth and may be much shorter than that of the fastest-growing normal mode which must ultimately dominate the solution as $t \rightarrow \infty$.

There are some unrealistic features inherent in the Eady model which are reflected in this example. The most obvious is the symmetry of the waves on the upper and lower boundary. Experiments with the variation of the Coriolis parameter included show that the

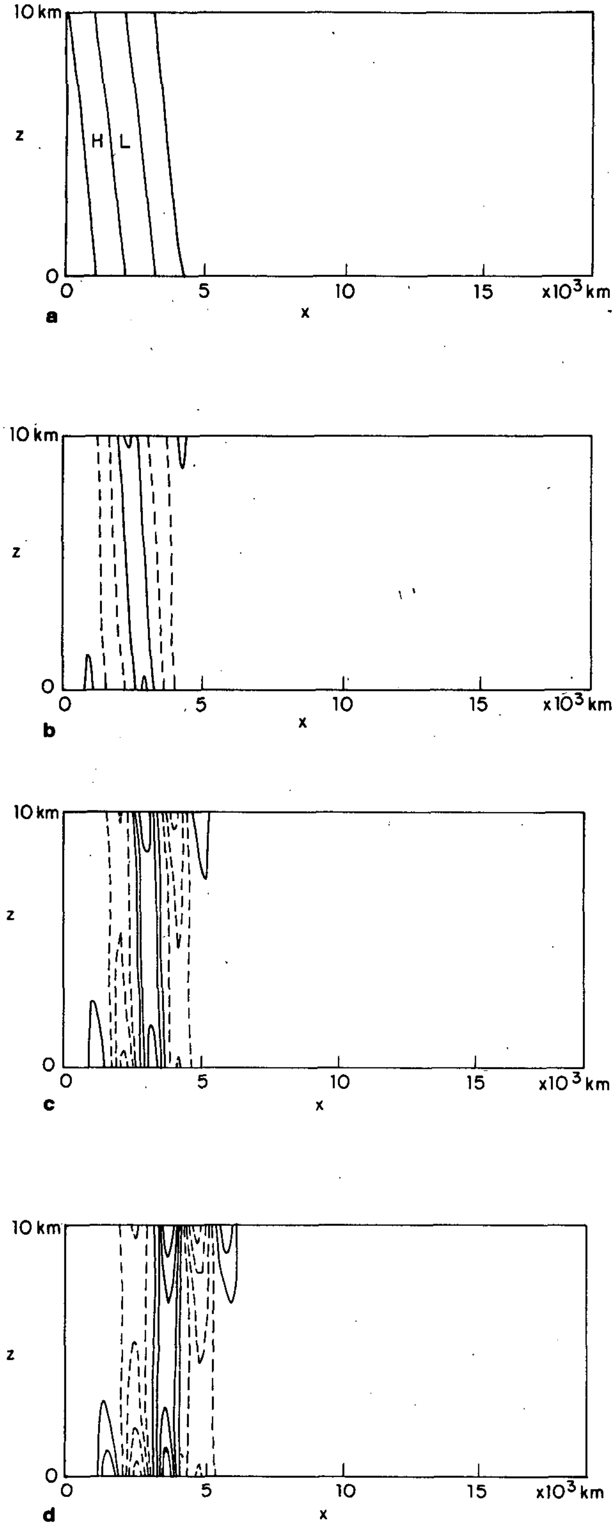

FIG. 1. Evolution of the initial condition streamfunction which favors cyclogenesis. In (a) taken at $t = 0$, the perturbation is outlined by the zero contour. This contour is suppressed in (b) taken at $t = 1.0$, (c) at $t = 2.0$, and (d) at $t = 4.0$. Negative contours are dashed and the interval between contours is 1.0.

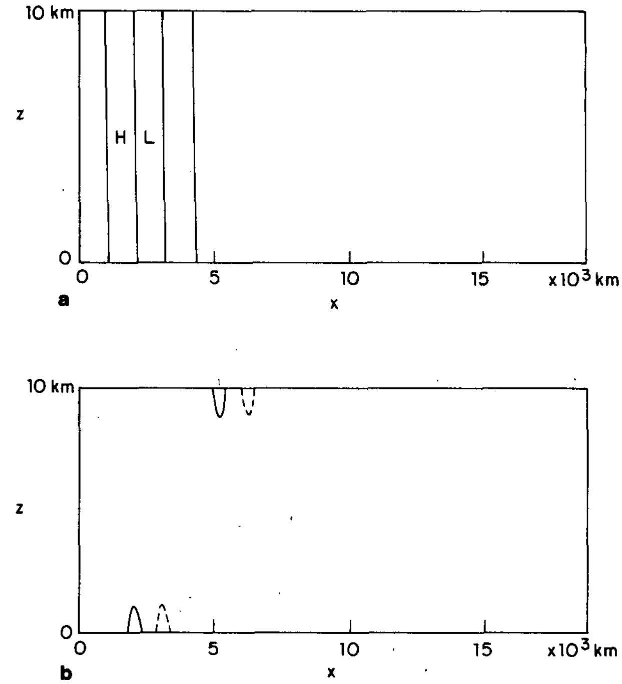

FIG. 2. As in Fig. 1, except the initial condition is barotropic, a configuration not favorable to cyclogenesis. In (a), the zero contour outlines the $t = 0$ state. Relatively little deepening is evident in (b), taken at $t = 4.0$.

upper wave becomes weaker and that the lower boundary wave is the relevant part of the solution (F1).

6. Conclusions

The interaction of the modal and non-modal waves with the mean flow in the initial value problem has been examined for the Eady model of baroclinic instability. Where neutral modes exist, these are excited providing a source of energy to what would otherwise be non-growing waves. For both exponentially unstable and neutral waves, the approximate time for setting up the modal solution is nondimensionally equal to the ratio of the vertical to horizontal wavenumber. In all cases the long-time asymptotic solution is dominated by the normal mode, the finestructure of the initial condition being lost (Pedlosky, 1964). While the long-time asymptotic for values of k for which unstable modes exist is dominated by the exponential growth, the setting up of the mode is dependent on the interaction among the continuous spectrum, the discrete modes and the mean flow. For values of k at which only neutral waves exist, the asymptotic amplitude of the normal mode is determined by this interaction.

An application of these ideas to a model of the early stages of cyclogenesis explains a specific observed feature: the relation between the upper-level trough and the developing cyclone. More generally, the observed atmospheric streamfunction may be resolved into a part of normal mode form and another which is non-modal. The modal waves may grow or decay but retain their structure while the non-modal waves continuously change structure and amplitude. Together these allow the great variety of observed motions despite the generally impoverished normal mode spectrum. One example of this variety is the rapid extraction of energy from the zonal flow and projection on a normal mode during explosive cyclogenesis.

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